DYNAMIC METEOROLOGY AND HYDROGRAPHY

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PART II.—KINEMATICS

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DYNAMIC METEOROLOGY AND HYDROGRAPHY

PART II. KINEMATICS.

BY

V. BJERKNES, TH. HESSELBERG AND O. DEVIK

CHAPTER I.

GENERAL CONSIDERATIONS ON THE OBJECT AND THE METHODS OF DYNAMIC METEOROLOGY AND HYDROGRAPHY.

87. The General Problem.—Treating statics of atmosphere and of hydrosphere we have considered invariable states of these media. Although passing occasionally the strict limits of statics, we never considered the states from the point of view of their variations, time never entering into our equations. But in entering upon the investigation of these states, not only from the point of view of their distribution in space, but also from that of their variation in time, we have to introduce *time* as a new independent variable. This allows us to view our problem in its generality and it will be useful to do this before returning to investigations of detail.

Considering the problem from a mathematical point of view, we have first to define our independent and our dependent variables.

We consider meteorological and hydrographic phenomena in relation to space and time, *i. e.*, our independent variables are *coordinates* and *time*. The system of coordinates is always rigidly attached to the earth. Two of the coordinates are the geographical ones, serving to define points at the surface of the land or of the sea; while the third has to give the height above or the depth below sea-level. In our static investigations we have found it convenient to measure this third coordinate in dynamic instead of in geometrical measure, and this will generally be convenient during the continued work.

As dependent variables we have to introduce the quantities required for defining the state of the atmosphere and the hydrosphere, or formulating the laws of the changes of these states. We shall designate these dependent variables as meteorological or hydrographic *elements*. The distribution in space of any of these elements is called its *field*. For the description of atmospheric states we have to consider at least five fields, those of pressure, of mass, of temperature, of humidity, and of motion. The first four of these are scalar fields; the fifth, that of motion, is a vector-field. The question may be raised if the full description of atmospheric states and of the laws of their changes will not require the introduction of still more fields. Thus there may be a mutual dependency upon one another of the meteorological processes and the electric or the magnetic fields of the earth. This would require the introduction of vectors describing these fields as further meteorological elements. But the rational plan will be, first, to treat the problem, as far as possible, with the smallest number of variables. We therefore restrict ourselves to the consideration of the five fields already defined for the case of the atmosphere. The five corresponding fields for describing the states of the hydrosphere and for formulating the laws of their changes are the fields of pressure, of mass, of temperature, of salinity, and of motion, precisely the same as in the case of the atmosphere, except that salinity takes the place of humidity.

The fields of pressure, of temperature, of humidity, and of salinity are described by the values of the corresponding elements observed in the different points of space. The fields of mass can be described in either of two ways, by the mass per unit volume or by the volume of unit masses. That is, we can consider either *density* or *specific volume* as the scalar element describing this field. In the same way we can use two different elements of vector-nature for describing the field of motion, either *velocity* or *specific momentum* (Statics, section 3).

Having defined our variables, we can thus concisely state the problem of meteorology and hydrography: *To investigate the five meteorological and the five hydrographic elements as functions of coordinates and time.*

88. Investigation of Phenomena Depending upon More Variables.—The general principle for investigating phenomena depending upon more variables is this: systematically to keep constant a certain variable or group of variables, in order to examine the effect of varying another variable or group of variables.

We have used this principle in statics already. Independent variables were then only the three coordinates. Among them the two geographical ones evidently form a natural group, having other relations to the investigated fields than the third coordinate, height. This difference determined the method. We began by considering the conditions of equilibrium along certain vertical (or quasi-vertical) lines, namely, the lines along which meteorological ascents or hydrographic soundings had taken place (Statics, Chapters VI and VIII); or in mathematical language, we gave to the geographical coordinates the constant values defining the stations and examined the effect of varying the third variable, height.

Using the results thus obtained, we afterwards drew synoptical charts, representing the fields by horizontal sections instead of by vertical soundings (Statics, Chapters VII and IX). This representation involves a modified use of the same general principle; for a chart shows the effect of varying the two geographical coordinates, while the third independent variable keeps constant.

When performing investigations according to this general principle it is occasionally convenient to let a certain dependent and a certain independent variable change parts. In this way we interchanged pressure and height. Retaining height as the third independent variable, to which the constant values were given, we arrived at isobaric charts drawn in level surfaces (section 65). Using pressure as the third independent variable to which the constant values were given, we arrived at topographic charts of isobaric surfaces (section 64). But in both cases the general result was the same, namely, a representation of the field of pressure in its relation to space, *i. e.*, in reference to coordinates as independent variables.

Introducing now a fourth independent variable, time, besides the three old ones, the coordinates, we have to apply the same general principle. The first question will then be that of the grouping of the variables. About this question there can be no doubt; for evidently the three coordinates form a natural group, having other relations to the phenomena than the fourth variable, time. The grouping of the variables being agreed upon, we can proceed along two ways: (1) Giving constant

values to the coordinates, we can examine the effect of letting time vary; or (2) giving a constant value to time, we can examine the effect of letting coordinates vary. These two different ways lead to two essentially different branches of meteorological and of hydrographic science.

89. Climatological Method.—First let us give constant values to the coordinates, and examine the effect of letting time vary. We can imagine the investigation performed in the following way: Self-recording instruments are set up at a number of fixed points (stations) in atmosphere or hydrosphere. The different records of the meteorological or hydrographic elements then show directly the effect of letting time vary, while the coordinates have the constant values defining a certain station.

When we examine the records we find great irregular changes, the explanation of which can not be found by a direct examination of the curves; but conspicuous signs of *regular* changes are also discovered. Forming averages in different ways, the irregular phenomena will more or less disappear. The regular ones will then, for the most part, present a periodical character, having the periods of the solar day, of the solar year, of the sunspots, and perhaps of still other cosmic phenomena. Besides the decidedly periodic phenomena, slow secular changes may also be discovered.

The different kinds of averages thus formed of the meteorological or hydrographic elements may be called the *climatological* elements for atmosphere or hydrosphere. Inasmuch as time enters into the definition of these elements, it is the local time of each station, not universal simultaneous time. The elements found at the different stations may be compared to each other. This leads to the drawing of climatological maps, showing the average influence of geographical data, just as the single curves showed that of astronomical events; but no way leads to the investigation of the nature or the causes of what we called irregular phenomena. These were eliminated, and to investigate them we must follow another way.

90. Dynamic Method.—In order to examine the other method, we can start with the records obtained from the same set of self-recording instruments, but shall make a modified use of them. Giving time a certain constant value, we read off from all records the values of meteorological or hydrographic elements at this epoch, and draw continuous synoptical representations of the field of each element. Having thus got a complete picture of the state of the atmosphere or the hydrosphere at this epoch, we give time a new constant value, read off the new values of the elements, and produce new synoptical representations of the fields, which give a complete picture of the state of atmosphere or hydrosphere at this second epoch, and so on.

A series of such pictures being produced, the next step will be to make them the subject of a comparative investigation. This comparative investigation of the successive states must lead to the solution of the ultimate problem of meteorological or hydrographic science, viz, that of discovering the laws according to which an atmospheric or hydrospheric state develops out of the preceding one.

We shall call this the *dynamic* method; for in virtue of the laws of hydrodynamics and thermodynamics which govern atmospheric or hydrospheric phenomena, preceding states are in relation of causality to subsequent states. Inasmuch as we know the laws of hydrodymanics and thermodynamics, we know the intrinsic laws according to which the subsequent states develop out of the preceding ones. We are therefore entitled to consider the ultimate problem of meteorological and hydrographic science, that of the precalculation of future states, as one of which we already possess the *implicit* solution, and we have full reason to believe that we shall succeed in making this solution an *explicit* one according as we succeed in finding the methods of making full practical use of the laws of hydrodynamics and thermodynamics.

91. Three Partial Problems.—Evidently general investigations according to the dynamic plan must lead to occupation with three special problems. The first is the question of the *organization of observations* serving these investigations. The observations being given, the next problem will be to work out from them synoptical representations of the fields serving to define actual states of atmosphere or hydrosphere. Introducing a terminology taken from medical science, we shall call this the problem of *diagnosis* of atmospheric or hydrospheric states. The result of a diagnosis being given, the final problem will be that of precalculating future states. Making continued use of the same terminology, we shall call this the problem of *prognosis* of future states. Before returning to details, we shall make some general remarks on each of these three problems, taking as the leading idea that the condition for real progress is to arrange so that full use can be made of the knowledge contained in the laws of hydrodynamics and thermodynamics.

92. Principles for the Organization of Observations.—It is of course not possible to know how observations will be organized later, when the problems of diagnosis and of prognosis are completely solved in explicit form. But the question interesting the present generation of investigators is to get that organization which would facilitate as much as possible the work with the solution of these problems.

From what we have evolved already it will be clear that the dynamic method requires simultaneous observations. The *principle of simultaneity* being therefore agreed upon as the fundamental one, the next questions will be those of the *distribution in space* of each set of simultaneous observations and the *distribution in time* of the successive epochs of observations.

In order to answer these questions, we have to remark that the fundamental laws of hydrodynamics and thermodynamics have the form of partial differential equations giving relations between the continuous space-variations and time-variations of the different elements. To make it as easy as possible to bring them into application, we must try to organize observations so as to realize an approximation toward *continuity* in space and time. In other words, the distances in space between the points of observation and the distances in time between the epochs of observation must be small enough to be used, with a certain degree of approximation, as line-differentials and time-differentials.

The test that the distribution in space of the points of observation fulfil this condition will be, that it turns out to be possible to draw synoptical maps, by use of the observations; for such maps give *continuous* representations of the fields of the observed elements. The distances to be allowed in the net of observations will therefore depend upon the space-variations of the elements. The network must be satisfactory for the element having the strongest space-variations. But nothing hinders elements which have less irregular distribution in space from being observed at a smaller number of points in the network.

A suitable time-differential must be determined by a comparison of synoptic charts representing the field of the same element at successive epochs. The changes which the element has undergone from epoch to epoch must be small enough to allow us to form satisfactory approximate values of the time-derivative of the element. The time-differential must therefore be chosen so as to suit the element which has the most rapid time-variation. But nothing hinders elements having slower time-variations from being observed, only, for instance, at every second or every third of the epochs of observation, which have thus been chosen.

93. **Special Remarks on Meteorological Observations.**—In passing to concrete meteorological observations, we shall first make some remarks regarding the *principle of simultaneity*.

The ideal is of course the use of self-recording instruments having sufficiently large time-scale. But whichever instruments or methods of observation be used, it will be neither possible nor required to realize simultaneity in the mathematical sense of the word. Most meteorological elements will under ordinary circumstances change very little during as small an interval of time as, for instance, half an hour. Departures of this magnitude from the precise epoch of observation will therefore not usually produce errors of greater importance, though exceptions are not excluded,

The general slowness of the variations makes it possible to use averages registered during suitable intervals of time instead of true instantaneous values. For one element, wind, the use of averages, as we shall see, will be unavoidable, and it will have certain advantages also in connection with other elements, especially inasmuch as time-integrations should be performed afterwards. But if averages be used, they should be used at all the cooperating stations, and taken according to the same rules at all. These conditions have been excellently fulfilled by hourly averages which we have obtained from the U. S. Weather Bureau.

Observations obtained from the higher strata by meteorological ascents will cause certain difficulties inasmuch as the records taken by the same instrument at different levels are not taken simultaneously. But the departures will be reduced according as we increase the velocity of the ascent. A registering balloon can be made to mount from the ground to the lower limits of the isothermal layer in less than an hour. Departures up to half an hour from the true epochs of observation being considered allowable, we are entitled to consider the observations obtained by such a balloon in different levels as simultaneous with observations taken near the ground half an hour after its launching.

Thus a tolerably satisfactory simultaneity *can* be realized even for the observations from the higher strata. But still the principle of simultaneity is not carried through universally, not even for the observations at the ground, where its realization should not cause any real difficulty. Thus departures by far exceeding the half-hour limit exist still in the European net of daily observations. Fortunately in the United States the principle of simultaneity is completely carried through for the whole net of stations. This circumstance, in connection with the complete homogeneity of the observations, all being obtained from self-recording instruments of the same construction and treated according to the same rules, make these observations the best which we have had at our disposal for the study of the conditions of the atmosphere near the ground.

Passing to the *distribution in space* of the points of observation, we must distinguish the points of observation near the ground from those in the free atmosphere. As to the investigation of the lowest atmospheric sheet, the greater nets of observation, as that of Europe, of the United States, or of India, may be said to be satisfactory, exceptions being made for certain specially difficult regions, for instance the western mountainous parts of the United States. For practical reasons the net of stations is here less close, while the space-variations of meteorological elements are stronger than in the flat land. For the most variable element, wind, this has caused us great difficulties.

In the free space fixed points of observation can not be maintained, and would not, unless they could be kept up in great number, be of appreciable use; for the lengths which can be used as line-differentials in vertical direction are much smaller than those which can be used in horizontal direction. But on account of the relative slowness of the variations in time and the rapidity with which meteorological ascents can be performed, we can get continuous records along vertical lines, representing approximately the instantaneous state of things along these lines.

As the variation of meteorological elements in horizontal direction is necessarily much smaller in the free atmosphere than near the ground, where the local influences of topography come in, it will not be necessary to provide all stations at the ground with the implements for meteorological ascents. But only experience can show how close the net of aerological stations should be. Further, it will not be required to give all aerological stations equally complete equipment, for the scalar elements have much less pronounced space-variations than the vector-element, velocity. As air-velocity is also much easier to observe, thanks to the method of pilot-balloons, it will be rational and economical to erect two classes of aerological stations, complete aerological stations and pilot-balloon stations. How close the net of each kind should be, will be evident by and by from the synoptical maps drawn by use of the ascents. The erection of aerological stations, including pilot-balloon stations in great numbers, will be of special importance in mountainous regions, where the effectivity of the common stations is so limited on account of the local irregularities.

The last and most delicate question is that of the determination of a suitable *time-differential* separating the epochs of observation. Inasmuch as continuity in time is realized in as great extent as possible by providing the stations at the ground

with self-recording instruments, the question will be reduced to that of a suitable interval between the successive aerological soundings. As time-variations of the meteorological elements have the same rapidity near the ground as in the free air, this question can be answered by examination of charts for the ground concerning the element which has the most rapid time-variations, namely, velocity. According to our preliminary experience regarding these charts (see Chapters XII and XIII) it seems reasonable to try time-differentials of three hours for this element, while differentials of double the length may be used for the other elements.

Observations of the completeness thus required can not be kept up continuously. It will be necessary to organize special periods of investigation extended for each time over a series of days. An effective organization of such a period would be this:

During the whole period continuous observations or observations for every hour of Greenwich time are kept up at all stations at the ground.

For every third hour of Greenwich time ascents are made from the pilot-balloon stations.

For every sixth hour of Greenwich time ascents are made from the complete aerological stations.

94. **Remarks on Hydrographic Observations.**—Oceanographic observations are not yet organized systematically. But the general principles for their organization will be the same as for the meteorological observations. Hydrographic expeditions going out occasionally can only contribute to the knowledge of the average state, *i. e.*, to the climatology of the sea. But the final aim must be that of investigating the actual states and their variations. The organization must then be governed by the principle of simultaneity. The investigations will have to be performed not by one luxuriously fitted ship, passing months or years at sea, but by the cooperation of small ships going out simultaneously.

The demands regarding the degree of simultaneity and the intervals between the epochs of observation will depend upon the rapidity of the changes. There are indications both for rapid changes (among which the tidal phenomena in the deeper strata will play an important part) as well as for slow seasonal changes and changes from year to year. The problem will be to organize observations so as to separate from each other the changes of different rapidity and to investigate them as much as possible independently of each other. But a serious discussion on the suitable method of organization will only be possible by and by, as our knowledge of the oceanic phenomena advances.

95. **The Problem of Diagnosis.**—The observations being given, the diagnosis will consist in working out continuous synoptical representations of the field of each element. This involves first the choice of proper methods of representing each field synoptically. This choice being made, methods for passing from the observations to the synoptic representation must be worked out. These diagnostic methods must take into consideration not only the observations themselves, but also all intrinsic relations existing between observed quantities and quantities to be represented. It is due to these intrinsic relations that we are able to work out relatively complete

representations in spite of the extreme incompleteness of the observations. According as we introduce the different relations of dynamics and thermodynamics, we shall have to examine carefully their possible diagnostic use.

In statics our work was exclusively of this diagnostic nature. We chose our methods for representing two fields, those of pressure and of mass, and we developed the methods of arriving at these representations, making diagnostic use of two relations, viz, the equation of hydrostatics and the gas-equation, respectively the relation existing between temperature, salinity, pressure, and specific volume of the sea-water. Passing now to kinematics, we shall have to occupy ourselves with the diagnosis of the field of motion. We shall choose methods for representing this field, and try to make complete diagnostic use of all relations of kinematic origin.

96. The Problem of Prognosis.—The present state being diagnosticated, the final problem is that of the precalculation of future states. The solution of this problem will involve the simultaneous use of all intrinsic relations of hydrodynamic and thermodynamic origin, to be used in connection with the initial conditions, the surface conditions, and data regarding exterior effects of terrestrial or cosmic origin. Evidently the problem is of enormous complexity. But in order to try to prepare its solution, we shall solve one by one a series of partial problems belonging to it. For every equation introduced we shall examine its prognostic as well as its diagnostic value. In kinematics we shall meet with the first partial problem of prognosis, for the definition of the fundamental kinematic vectors involves the idea of time. When we know the instantaneous velocity of a moving particle, we shall know the place of this particle a differential of time later. The changes of place of the moving particles can therefore be determined in the first approximation by purely kinematic principles. The solution of this problem of kinematic prognosis is the first step in the solution of the general problem.

During the work with the problem of prognosis, it will be apparent that while we are probably in possession of all the intrinsic relations to be used for its solution, certain empirical data required for bringing them into application must be sought. The missing data can in many cases be found by reversing the problem of prognosis. The state being known at two epochs, we calculate the missing data, which, used in the intrinsic relations, should allow us to calculate the second state when the first is given. Having this method in view, we shall treat the different partial problems of prognosis both in direct and in inverse form.

Reversing the problem of kinematic prognosis, we shall thus arrive at the purely kinematic determination of accelerations. When we determine afterwards the same accelerations by dynamic principles, we get the opportunity of finding the value of a term in the dynamic equation, of which we have not *a priori* a sufficient knowledge, namely, of that representing frictional resistance.

CHAPTER II.

THE OBSERVATIONS OF AIR AND SEA MOTIONS.

97. **The Common Wind-Observations.**—Taking up the subject of the kinematics of atmosphere and hydrosphere, we have first to discuss the observations to be used as the basis of the kinematic diagnosis. We shall begin by considering the observations of wind.

Even a rough examination shows the wind to be very irregular, its direction and intensity changing rapidly in varying limits. By using finer methods of observation smaller irregular air-movements will be discovered which would otherwise escape our attention. Directions and intensities of wind noted at meterological stations are therefore always averages, the smaller irregularities not being discovered and the greater ones being smoothed out by the personal estimate of the observer or by a regular treatment of the records of the self-recording instruments.

It is therefore only certain average air-motions which can be submitted to a kinematic analysis. Neglecting the small irregularities in the large-scale meteorology, we make a similar approximation as when in laboratory experiments on fluid motion we neglect the irregular molecular motions existing according to the kinetic theory. But in both cases indirect effects of the small motion arise in the form of an apparent increase of frictional resistance. The question of this resistance will be taken up in the dynamic part of this book.

For our kinematic investigations we have to mention these irregularities only on account of the uncertainty which they cause in the noted average direction and intensity of the wind. When quantitative use is to be made of the wind-observations, it will be important to use rational methods both for taking the observations and for smoothing out the irregularities. Especially it will be important that the *same* method should be used for these purposes at all cooperating stations. The best results will be obtained by self-recording instruments, the averages being taken from the values registered during an interval of time extended equally long before and after the epoch of observation. The average should be formed by *vector-addition* and registering instruments should allow an easy determination of this average. The vector formed by taking the separate averages of the recorded directions and of the recorded intensities will not be the true vector-average; but it may be used approximately instead of the true vector-average if the variations of direction and intensity have not been too strong during the interval for which the average is formed. As meteorological wind-observations have not been organized in view of our quantitative applications, they are very imperfect from our point of view. In Europe, besides the fundamental imperfection that the principle of simultaneity is not carried through, all sorts of wind-observations are used, from personal estimates

to averages obtained by the best self-recording instruments. Greater homogeneity is highly desirable. The best wind-observations which we have had at our disposal are from the United States. We have obtained them partly from the published weather maps, and partly from unpublished material, thanks to the kindness of the United States Weather Bureau. They give the registered average wind-velocities from hour to hour, and eight corresponding average wind-directions. We have considered these averages as defining the vector-average of the wind for the half-hour epoch, though for periods of rapid changes they may differ considerably from the true vector-average. Quite independently of the method of averaging, it would be a great improvement to increase the number of wind-directions noted from eight to at least sixteen.

98. Preliminary Synoptic Representation of the Wind-Observations.—A set of simultaneous observations of the wind being given, the first step in the subsequent diagnostic work will be to introduce these observations in a convenient form on the map.

The most direct way will be to draw on the map a set of arrows representing the observed wind-directions, and to add numbers representing the observed wind-velocities in meters per second. These numbers, giving the result in quantitative form, should always be introduced instead of the different qualitative signs used to represent the strength of wind according to the different "wind-scales." Plates XXXI, XXXVI, and LIII give examples of charts containing in this way a representation of wind-observations.

Besides representing the directions by arrows, we shall use a method of representing them by numbers. This will be useful not only for purposes of registration, but also for quantitative work. The correspondence between directions and numbers which we shall use is illustrated by fig. 32.

The numbers defined by this figure may be used not only as names of the directions, but also as measure of the angles which the different directions form with the initial direction, the direction toward E. We get in this manner a measure of the angles by dividing the circle into 64 instead of 360 degrees. We have chosen this measure of angles for our purposes by two reasons; first, 64 is the highest two-figure number which is a power of 2; and then its tenth part, 6.4, differs only by 1.9 per cent from 2π or 6.28. This difference will as a rule be insignificant for us. We can therefore consider the numbers 1 to 64, after division by 10, as representing the angles in absolute measure. The choice of the direction E. N. W. S., *i. e.*, the direction against the motion of the hands of a watch, as the positive, and the direction toward E. as the initial direction, is made for reasons which will be apparent later, when we shall choose our systems of coordinates and give the corresponding rule of signs.

When this correspondence between numbers and directions is used, it will be found convenient to have the diagram of fig. 32 engraved on a transparent sheet of glass or of celluloid. By use of this divided plate we can then easily pass from an arrow to the corresponding number, or vice versa.

These numbers can now be introduced on the charts instead of the arrows. The chart will then contain a representation of the wind-observations by use of two sets of numbers, one set representing the wind-directions and another representing the wind-intensities. It will be found convenient to use ciphers of different type or of different color for the two different sets. The heavy numbers on plates XXXVI and LIII represent wind-directions.

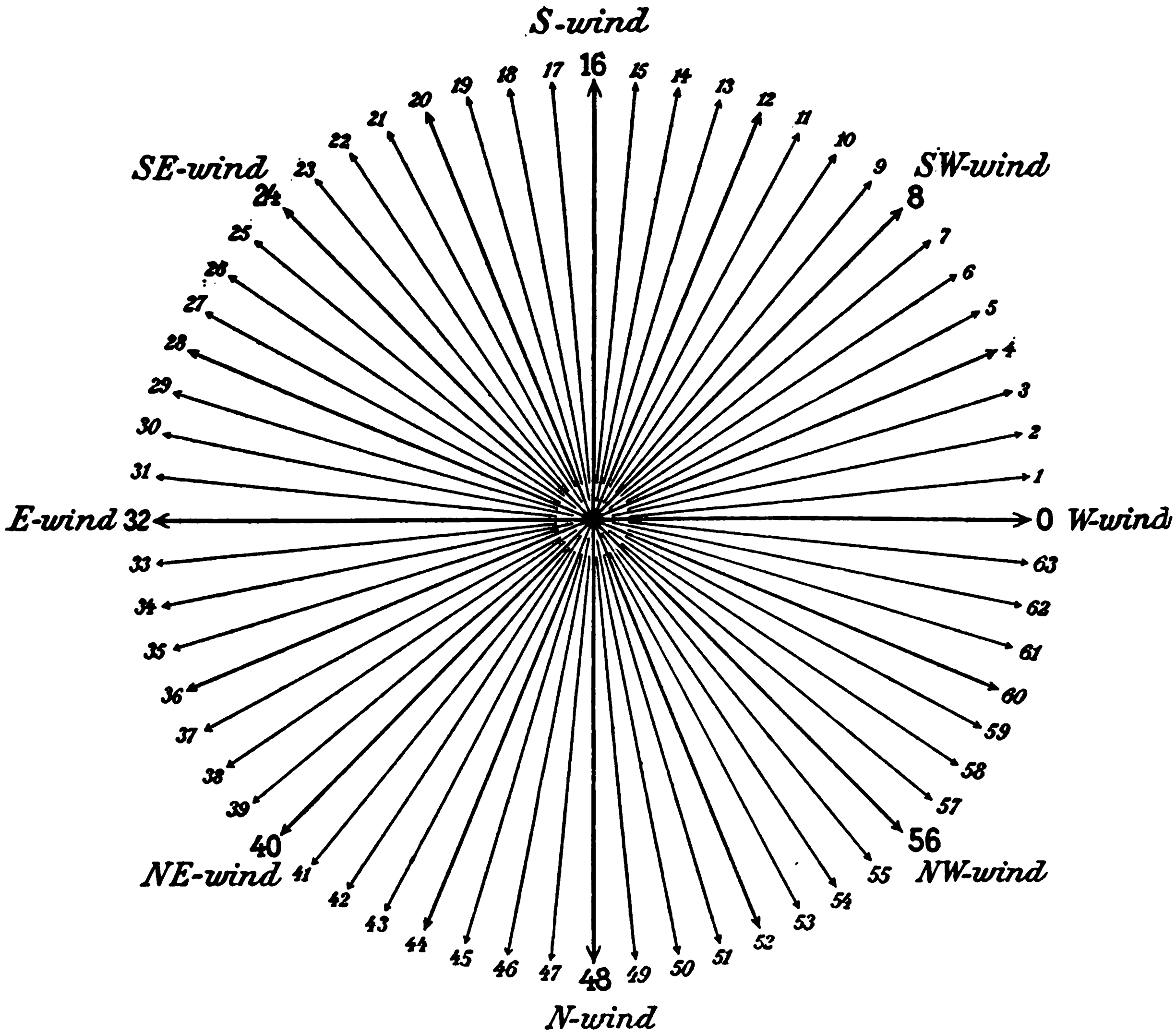

FIG. 32.—Representation of directions by numbers.

Charts of this description, which contain a representation of the observed motions either by arrows and numbers or by two sets of numbers, will form the starting-point for the whole subsequent work of kinematic diagnosis.

99. Observations of Air-Motion in the Free Atmosphere.—Until lately the drift of the clouds gave the only information on air-motion in the higher strata. Qualitative results were obtained by observing the cloud-form and the direction of the drift, and quantitative results in the first approximation by also measuring the

angular velocity of the cloud by a nephoscope. For as the height of the cloud can be estimated approximately by its form, the velocity can be calculated. But besides the smaller errors caused by the use of the estimated heights, great errors may arise on account of the difficulty of recognizing with perfect certainty the cloud-forms. Even if finer methods be used, based upon the measurement of the parallax, the diffuseness of the objects observed causes difficulties, and the process of formation and dilution of clouds going on simultaneously with their motion makes the interpretation of the observations difficult.

But the great drawback of the cloud-observations is that they only give sporadic information, depending upon where clouds happen to be. What is wanted are *continuous records* of the air-motion taken along vertical or quasi-vertical lines, corresponding to the continuous records of pressure, temperature, and humidity, which we have considered in Statics. Continuous records of air-motion may be obtained by the same ascents which give records of the scalar meteorologic elements. Besides the other instruments, a kite can lift an anemometer registering the wind-intensity, while the direction can be estimated roughly by the direction of the kite-line. But the best results are obtained by observing the motion of free balloons by theodolites. If the height of the balloon can be found from other data, only one theodolite is required. Thus if the balloon carries a registering barograph, the direction and intensity of the wind may be found as function of the registered pressure or as function of the height calculated from this pressure. But an important simplification has recently been introduced. A closed caoutchouc-balloon has been found to mount with a practically constant velocity, which can be calculated by the dimensions and buoyancy of the balloon. Thus the height is known simply by the time elapsed since the moment of its launching. The air-motion can therefore be determined much more easily than all other meteorological elements in the free air, the instruments required being simply a theodolite and a small pilot-balloon. According to Hergesell* this method gives better determinations of the air-motion in the higher strata than our ordinary station-instruments can give for the layer near the ground.

Just as other instruments, the pilot-balloons give the air-motion with the small irregularities to some degree smoothed out. Small oscillations of the balloon are seen as long as the distance is not too great, but as the observations are taken at intervals which are long compared to the period of these small irregularities, only averages are obtained.

It should be observed that these averages in reality are of a complex nature, being averages simultaneously as regards intervals of time and of height, and further, that the air-motions found by the same pilot-balloon in different heights are not strictly simultaneous. We have mentioned already this general imperfection of observations obtained by aerological ascents (section 93) and its relatively small importance when the ascents are arranged so as to be made with sufficient vertical velocity.

*H. Hergesell: Die Bedeutung der Pilotballonaufstiege für die praktische Aerologie. Sixième Réunion de la Commission Internationale pour l'Aérostation scientifique à Monaco, 1909. Strassbourg, 1910.

100. Horizontal Motion and Vertical Motion.—The considered observations, those from the earth's surface as well as those from the higher strata, do not give full information on the direction of the motion. They only give the azimuth of the direction, not its inclination relatively to the horizon. Observations of the vertical components of the motion are difficult. It has been proved possible lately to derive the vertical velocity of the air from the motion of pilot-balloons, the observations being taken in a more complete way by two theodolites and base.* This or other methods of making the observations more complete are very much to be recommended, especially also on account of the more correct values thus obtained for the horizontal velocity. But even if it be possible thus to obtain valuable results on the local ascending or descending currents, it may turn out difficult to arrange a sufficient number of observations for the purpose of getting a complete picture of the general vertical motion. As long as a sufficient system of observation of this nature has not been organized, we shall be obliged to derive the vertical motion indirectly. This can be done by proper diagnostic methods which will be developed later, provided that we know sufficiently well the horizontal motion. We shall therefore first examine this part of the motion as completely as possible.

101. Direct Result of a Pilot-Balloon Ascent.—Directing our attention to the horizontal motion only, we shall consider the result of the ascent of a viséed balloon. Table A, columns 1, 3, and 4, shows the result of an ascent as given in the publications of the International Committee for Aeronautical Meteorology.†

A table like this gives more detailed information on the air-motion than we can use in the subsequent work, when the result of a great number of simultaneous ascents are to be worked out. The contents of the table must therefore be condensed, and evidently by forming vector-averages of the air-motion for thicker sheets than those appearing in table A.

102. Vector-Averages of Horizontal Motion Formed with Height as Independent Variable.—As the required averages have to be found by vector-addition, a graphical method will be best. From table A we derive a curve giving a geometrical representation of the distribution of velocity in the different heights. We form the numbers noted in column 5, obtained as products of the velocities, column 4, into the thicknesses Δz of the corresponding sheets, column 2; drawing then in succession segments of line having the lengths represented by the numbers in column 5 and the directions given in column 3, we get a polygonal curve which is seen in each of the diagrams, figs. 33 and 34 (pages 15 and 17). The numbers added in the corners represent the heights.

Now let us mark on the curve two points, representing two heights, and let us draw the chord joining them. This chord then represents the vector-sum (or the vector-integral) of velocities within the sheet defined by the two points, formed with height as independent variable; and dividing by the thickness of the sheet we shall get the average velocity within this sheet. In each of the figures 33 and 34 a

*See note, p. 12.
†Publications de la Commission Internationale pour l'Aérostation scientifique, 1907, p. 358. Strassbourg, 1909.

set of such chords are seen, drawn to determine the average velocities in the corresponding sheets.

If we wish to have the air-motion represented by specific momenta instead of by velocities, the direct way of proceeding will be to change the velocities contained in column 4 of table A into specific momenta, multiplying them by the corresponding densities of the air. Afterwards the construction is performed exactly as in the case of the other vector, velocity. We multiply the numerical values of the specific momenta by the corresponding thicknesses of sheet Δz, and draw in succession segments of line having the directions given in column 3 and the lengths represented by these products. By use of the curve thus obtained we form the vector-average for any sheet precisely as in the case of velocities.

TABLE A.—*Horizontal velocity of viséed balloon in different sheets. Pavia (lat. 45° 11′, long. 9° 10′ E.), July 25, 1907, $7^h 33^m - 7^h 48^m$ a. m., Greenwich.*

1 Height z (meters).	2 Thickness of sheet Δz (meters).	3 Direction of motion within the sheet.	4 Velocity v within the sheet (m/sec.)	5 $v.\Delta z$.
77				
	603	S 50° E	3.4	2050
680				
	280	S 57° E	4.0	1120
960				
	280	S 36° E	5.3	1484
1240				
	290	S 28° W	1.5	435
1530				
	280	S 2° W	1.8	504
1810				
	280	S 2° W	2.0	560
2090				
	340	S 35° W	1.5	510
2430				
	300	S 53° W	1.8	540
2730				
	310	S 69° W	1.8	558
3040				
	360	S 55° W	3.0	1080
3400				
	310	S 53° W	2.8	868
3710				
	320	S 58° W	4.4	1408
4030				
	370	S 37° W	10.2	3774
4400				

103. The Choice of Suitable Atmospheric Sheets.—The method of forming the vector-averages being given, we have to settle the choice of sheets for which the averages should be formed in our practical work. Here different ways may be thought of.

One possibility will be to retain the principle used in table A, viz, to use a division into sheets characterized by the motion itself, only trying to reduce the number of sheets. This method will be very natural if corresponding changes of wind-direction and wind-intensity are found by the simultaneous ascents from other stations, and especially if these changes of motion are connected with changes of temperature or humidity.

We emphasize this as a method which may be seriously thought of, especially later, when more complete observations can be obtained. But for the first attempts we shall prefer a more summary method, using the same sheets which we have already introduced in statics for the representation of the fields of pressure and of mass. This will be convenient for several reasons. First, the kinematic diagnosis is not complete as long as we know only the velocities. We must also know the amounts of mass which have these velocities. Using the sheets introduced in

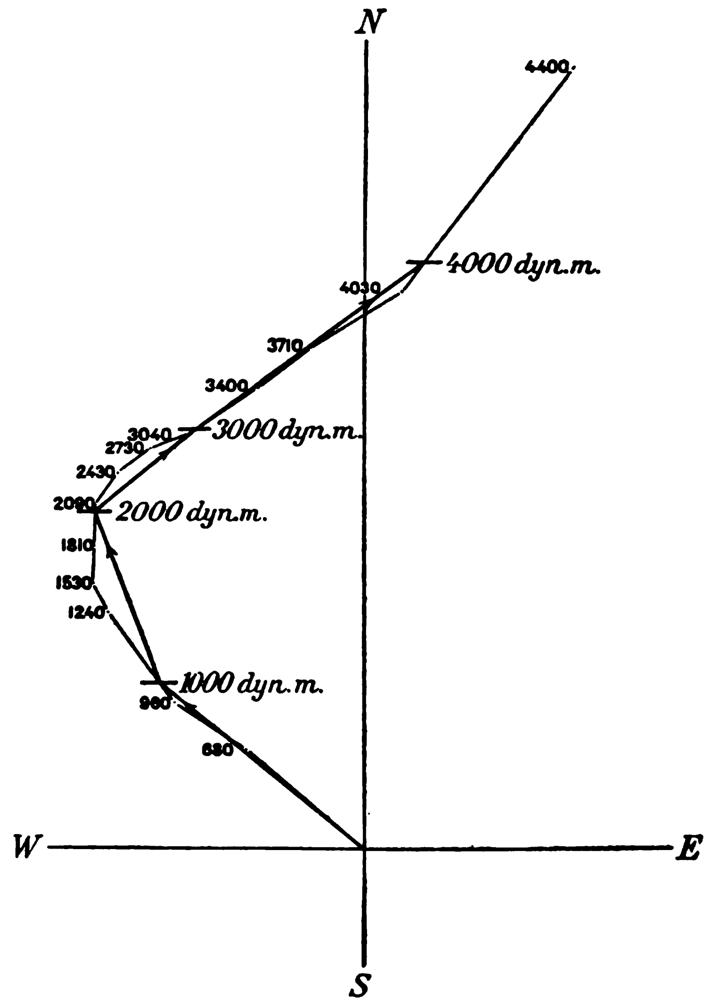

FIG. 33.—Construction of average velocities for standard level sheets.

statics, we get a coherent representation of velocities and of masses. Further, our final aim being the performance of dynamic investigations, we shall arrange everything convenient for future purposes by choosing our representation of the field of motion in as close connection as possible with that of the field of pressure. We shall therefore use either level sheets of the thickness of 1,000 dynamic meters, or isobaric sheets corresponding to the difference of pressure of 100 m-bars. In so doing, we shall evidently often get sheets which are too thick for a detailed representation of the motions. But the way of refining the representation by the choice of thinner

sheets is evident. Thus level sheets of 500 or of 100 dynamic meters, or isobaric sheets of 50 or of 10 m-bars may be used, especially in the lowest strata near the ground, where the greatest irregularities occur.

104. Use of Standard Level Sheets.—Fig. 33 shows the construction leading from the observations given in table A to the average velocities in standard level sheets. The curve having been constructed as described in section 102, points are marked on it corresponding to the heights of 1020, 2040, 3060, 4080 . . . meters, *i. e.*, to 1000, 2000, 3000, 4000 . . . dynamic meters. Then the chords are drawn, and the numbers representing their directions found by use of the divided sheet described in section 98. These numbers are given in column 4 of table B. Further, the length of the chords is measured and the numbers representing these lengths divided by 1020, which represents the common thickness of the level sheets. The velocities found in this way are given in column 5 of table B.

As the balloon carried self-recording instruments, the pressures in the standard level surfaces have been calculated (Statics, secs. 53–54) and are given in column 2

TABLE B.—*Average horizontal motion in standard level sheets. Pavia (lat. 45° 11', long. 9°10' E.), July 25, 1907, $7^h33^m-7^h48^m$ a. m., Greenwich.*

1 Height dyn. meters.	2 Pressure m-bars.	3 Density ton/m^3.	4 Direction of motion.	5 Velocity m/sec.	6 Sp. momentum 10^{-3}ton/m^{2}sec.
4000	622				
		0.00083	6	3.8	3.2
3000	705				
		0.00092	7	1.6	1.5
2000	797				
		0.00102	20	2.4	2.5
1000	899				
		0.00112	25	3.7	4.1
75	1003				

of table B. The difference between these pressures multiplied by 10^{-5} gives the average density of the air in the different level sheets (Statics, sections 38 and 54). These densities appear in column 3 and give full information on the masses moving with the velocities specified by the columns 4 and 5.

To get the corresponding specific momenta, we can simply multiply the average velocities, column 5, by the corresponding densities, column 3. The result is noted in column 6.

Regarding the specific momenta obtained in this way, it should, however, be emphasized, that they are not identical in direction and intensity with those average momenta which would be found if we multiplied each velocity given in table A by the corresponding density and repeated the construction for forming the vector-average. But on account of the relatively slow decrease of density with the height, the difference will generally be insignificant in comparison with the unavoidable uncertainty of the observations of velocity.

105. Use of Standard Isobaric Sheets.—Fig. 34 shows the construction leading from the observations given in table A to the average velocities in standard isobaric sheets. The curve is precisely the same as that of fig. 33. But now we

have to mark on it the points which represent the heights of the standard isobaric surfaces. From the records in the case before us these heights are found equal to 99, 989, 1970, 3057, 4274 dynamic meters, as noted in column 2 of table C. The corresponding thicknesses of sheet, also expressed in dynamic meters, are given in column 3. These two columns then represent, as we have seen (Statics, secs. 35, 54), the distribution of pressure and mass along the vertical, the thickness of the standard isobaric sheets giving the average specific volume of the air within the sheet.

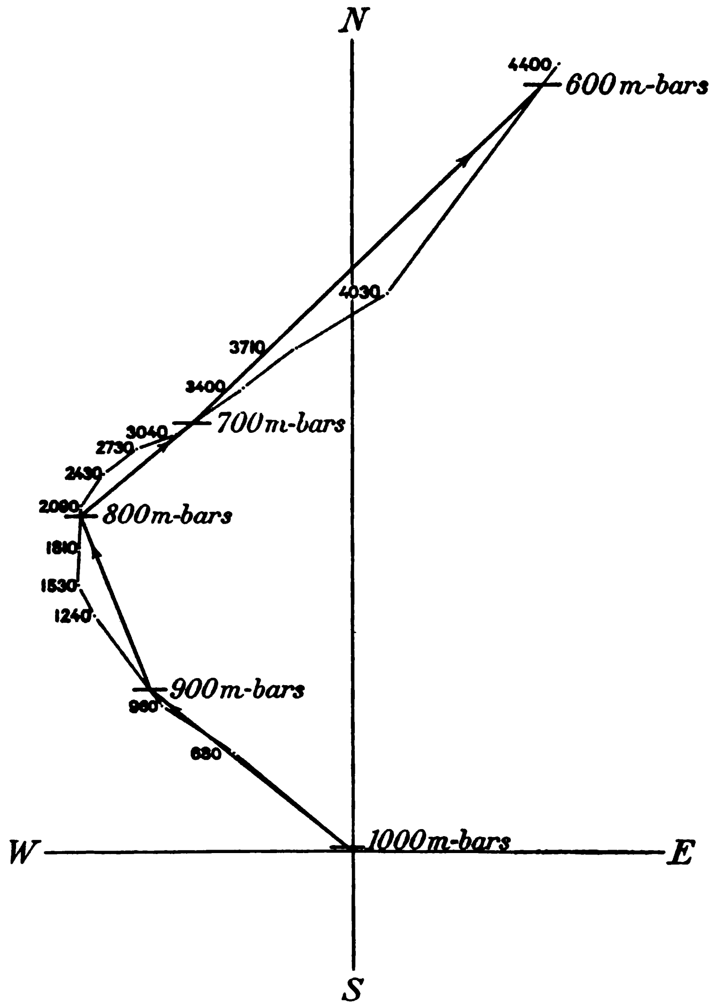

FIG. 34.—Construction of average velocities for standard isobaric sheets.

On the curve we now mark points representing the heights noted in column 2, *i.e.*, the heights of 101, 1009, 2009, 3121, 4362 common meters, and draw the corresponding chords. The directions of these chords determined by the transparent sheet (fig. 32) are noted in column 4. Then the lengths of these chords are measured and divided by the thicknesses of sheet, viz, 908, 1000, 1112, 1241 common meters, respectively. The average velocities found in this way are noted in column 5.

As the numbers in column 3 represent the average specific volume of the air in the sheets, we get the specific momenta simply by dividing the velocities (column 5)

by the specific volumes (column 3). The result is given in column 6. But as in the preceding case, we have to remark that the values of specific momentum found in this way are not the exact height-averages of specific momentum for these sheets; yet they give as a rule sufficient approximation toward these averages. If the exact values are required, we have to change the velocities given in table A into specific momenta before performing the construction leading to the averages.

TABLE C.—*Average horizontal motion in standard isobaric sheets. Pavia (lat. 45°11′, long. 9°10′E.), July 25, 1907, $7^h33^m-7^h48^m$ Greenwich.*

1 Pressure m-bars.	2 Height dyn. meters.	3 Sp. volume m³/ton.	4 Direction of motion.	5 Velocity m/sec.	6 Sp. momentum 10^{-3}ton/m² sec.
600	4274				
		1217	8	5.2	4.3
700	3057				
		1087	7	1.7	1.6
800	1970				
		981	20	2.4	2.4
900	989				
		890	25	3.7	4.2
1000	99				
		....	25	3.4	4.0
1002.6	76				

106. Special Remarks.—As we see, the result contained in columns 4 and 5 of table B, *i.e.*, the average velocities of the air in the level sheets, can be found without any knowledge of the registered values of pressure, temperature, and humidity. For everything regarding specific momenta or motions in isobaric sheets, a certain knowledge of the fields of pressure and of mass is required. But on account of the very limited accuracy of the observations of velocities we shall never, for the purely kinematic purposes, need to know these fields of pressure and of mass with the accuracy used in statics for the purpose of drawing their synoptical representations. Pressure and temperature being observed near the ground, and the values of the temperature being estimated for greater heights, it will be easy, by using our meteorological tables (11M, 15M, 12M, 10M), to find the heights of the standard isobaric surfaces or the pressures in standard levels and the correlated data regarding specific volume or density, with sufficient accuracy for kinematic purposes.

It is further worth while mentioning that the result of an ascent (as contained, for instance, in columns 4 and 5 of table C), may be condensed into a telegram which could be sent to a central office. (Compare Statics, section 57.) While two men are taking the observations, a third can perform the constructions and the calculations, and it will be possible to send off the telegram a few minutes after the last observation of the pilot-balloon is taken. Thus there will be no technical difficulty in bringing observations of this kind into application for the daily weather service.

107. Main Example: Europe, July 25, 1907.—The most complete observations which we have had at our disposal for working out a diagnosis of atmospheric motions are those obtained on July 25, 1907. This day belonged to a period of

six days, extending from July 22 to July 27, during which the North Pole was surrounded by a circle of aerological stations. During this period in all 89 registering balloons, 20 manned balloons, 100 kites and captive balloons, and 41 pilot-balloons were sent up.

From our point of view these observations were spread out over too great an area as well as over too long a period of time. The number of pilot-balloons launched was also far too small compared with that of registering balloons (compare sections 92, 93). In order to try a diagnosis of atmospheric motions we can only think of using the observations from a more limited area, where the network of stations was closest, namely, central Europe. And we shall choose the epoch when the greatest number of fairly simultaneous ascents took place, namely, July 25, about the time of the daily meteorological observations, 7 a. m. Greenwich. From one hour before to two hours after this epoch 13 balloons were followed by theodolites. This number is not sufficient for working out a real diagnosis of atmospheric motions over Central Europe, but we shall at least be able to illustrate the formal methods.

When working out the example we shall choose the method of dividing the atmosphere into isobaric sheets; the corresponding use of level sheets will be understood without difficulty. Using the methods developed in Statics, as well as those given in section 105, we get the result of the ascents condensed in table D. In each of the 13 subdivisions of this table, the first column gives the standard pressures and the pressure at the station; the next gives the dynamic height of these pressures, and the three following the thickness of sheets, direction, and velocity of air-motion in the sheets. It may be remarked regarding the observations that those of the wind in subdivision 8 are obtained from the ascent of a kite, those in subdivisions 11 and 12 from the course of manned balloons. The registering balloon, subdivision 13, could not be viséed for cloudiness. In subdivision 7, heights of standard surfaces and thickness of standard sheets are estimated from the ascents at the other stations.

From the numbers registered in table D we shall now work out the corresponding synoptical representations. Using the numbers representing the heights of the standard isobaric surfaces and the thickness of the sheets contained between them, we shall first work out representations of these sheets. For the sake of brevity we shall denote these sheets, counted from below, by the Roman numbers X, IX, VIII, . . . , X being the sheet limited below by the 1000 m-bar surface, IX that limited below by the 900 m-bar surface, and so on. The always incomplete sheet contained between the 1000 m-bar surface and the ground may be denoted by XI.

To distinguish the curves for absolute and those for relative topography we shall draw the first as single and the second as double lines. The double lines consist of a thick and a thin line, the thin being drawn on that side where the isobaric sheet, whose thickness is represented, is thinner. Fig. A of plate LVII represents the isobaric sheet X, the single lines giving the dynamic height of the 1000 m-bar surface above sea-level and the double lines giving the height of the 900 m-bar surface above the 1000 m-bar surface. Or, as we express it: the single lines give the absolute topography of the 1000 m-bar surface and the double lines the relative topography of the 900 m-bar surface. In the same manner fig. A of plate LVIII

TABLE D.—*Aerological observations* (*Europe, July 25, 1907*).

1. Uccle. Lat. 50° 48′ Long. 4° 22′ E. Dyn. height 98 6^h52^m–7^h45^m

100	16374		
	4427	0	4.7
200	11947		
	2627	10	3.2
300	9320		
	2019	18	3.4
400	7301		
	1653	19	3.3
500	5648		
	1408	8	2.6
600	4240		
	1220	5	2.5
700	3020		
	1082	4	2.5
800	1938		
	963	4	1.4
900	975		
	867	36	4.5
1000	107		
	—	35	4.5
1001.2	98		

2. Crinan. Lat. 56° 6′ Long. 5° 32′ W. Dyn. height 5 9^h20^m–10^h12^m

400	7187		
	1654		
500	5533		
	1374		
600	4159		
	1184		
700	2975		
	1052	9	3.4
800	1923		
	945	4	2.1
900	978		
	863	11	2.7
1000	115		
	—	4	4.0
1013.2	5		

3. Guadalajara. Lat. 40° 39′ Long. 3° 10′ W. Dyn. height 622 8^h29^m–9^h13^m

300	9578		
	2145	53	17.5
400	7433		
	1725	54	13.6
500	5708		
	1432	55	11.4
600	4276		
	1226	53	7.3
700	3150		
	1077	52	6.8
800	1973		
	966	52	4.1
900	1007		
	—	48	1.2
942.1	622		

4. Pavia. Lat. 45° 11′ Long. 9° 10′ E. Dyn. height 76 7^h33^m–8^h10^m

200	11889		
	2601		
300	9288		
	1993		
400	7295		
	1631		
500	5664		
	1390		
600	4274		
	1217	8	5.2
700	3057		
	1087	7	1.7
800	1970		
	981	20	2.4
900	989		
	890	25	3.7
1000	99		
	—	25	3.4
1002.6	76		

5. Zürich. Lat. 47° 23′ Long. 8° 33′ E. Dyn. height 471 7^h17^m–8^h3^m

100	16238		
	4421	3	10.0
200	11817		
	2577	6	6.5
300	9240		
	1992	7	7.6
400	7248		
	1622	2	10.2
500	5626		
	1382	3	6.7
600	4244		
	1206	2	6.8
700	3038		
	1083	62	5.3
800	1955		
	978	4	0.6
900	977		
	—	30	2.1
955.9	471		

6. Strassburg. Lat. 48° 35′ Long. 7° 45′ E. Dyn. height 138 8^h8^m–8^h42^m

300	9235		
	1982		
400	7253		
	1630		
500	5623		
	1385		
600	4238		
	1203	0	7.0
700	3035		
	1078	2	6.7
800	1957		
	974	8	2.0
900	983		
	—	37	2.1
995.9	138		

7. Hamburg. Lat. 53° 33′ Long. 9° 59′ E. Dyn. height 17 7^h48^m–ca. 8^h40^m

200	11890		
	2649	59	9.2
300	9241		
	2001	57	10.5
400	7240		
	1597	58	8.8
500	5643		
	1447	55	8.0
600	4196		
	1205	49	2.9
700	2991		
	1064	41	2.9
800	1927		
	946	38	1.9
900	981		
	863	56	4.3
1000	118		
	—	55	3.4
	17		

8. Lindenberg. Lat. 52° 36′ Long. 13° 37′ E. Dyn. height 116 6^h40^m–8^h58^m

200	11862		
	2624		
300	9238		
	1999		
400	7239		
	1644		
500	5595		
	1395		
600	4200		
	1210		
700	2990		
	1062		
800	1928		
	948	56	9.0
900	979		
	864	59	6.8
999.9	116		

9. München. Lat. 48° 9′ Long. 11° 37′ E. Dyn. height 506 7^h4^m–7^h34^m

200	11874		
	2579		
300	9295		
	2002		
400	7293		
	1641		
500	5652		
	1396		
600	4256		
	1215	62	10.0
700	3041		
	1082	62	10.0
800	1959		
	974		
900	985		
	—		
953.2	506		

10. Wien. Lat. 48° 15′ Long. 16° 21′ E. Dyn. height 157 7^h0^m–9^h40^m

500	5631		
	1394	60	17.3
600	4237		
	1213	61	14.3
700	3024		
	1072	62	15.3
800	1952		
	959	62	9.7
900	993		
	—	51	3.6
995.9	157		

11. Novogeorgievsk. Lat. 52° 26 Long. 20° 44′ E. Dyn. height 108 5^h37^m–9^h12^m

700	2947		
	1054	53	12.0
800	1893		
	947	54	8.4
900	946		
	—	57	3.0
997.2	108		

12. Koutchino. Lat. 55° 45′ Long. 37° 59′ E. Dyn. height 138 6^h38^m–7^h15^m

200	11566		
	2588		
300	8978		
	1939		
400	7039		
	1584		
500	5455		
	1345		
600	4110		
	1176		
700	2934		
	1043	60	7.5
800	1891		
	942	62	7.0
900	949		
	—	62	7.0
993.2	138		

13. Pawlowsk. Lat. 59° 41′ Long. 30° 29′ E. Dyn. height 29 5^h9^m–6^h8^m

300	8442		
	1920		
400	7022		
	1578		
500	5444		
	1346		
600	4098		
	1174		
700	2924		
	1046		
800	1878		
	946		
900	932		
	859		
1000	73		
	—		
1005.2	29		

represents the isobaric sheet IX, the single lines giving the absolute topography of the 900 m-bar surface and the double lines the relative topography of the 800 m-bar surface, and so on.

On the charts representing thus the different isobaric sheets, we now introduce the arrows and corresponding numbers representing the air-velocities given in table D. These data regarding the air-motions in the higher strata, in connection with the corresponding data for the ground which are given on plate LIII, will now form the basis for the further diagnostic work regarding the air-motion above central Europe, July 25, 1907, about 7 a. m. Greenwich.

108. On the Observations of the Sea-Motions.—If the observations of the air-motions are too scarce, this is still more the case with those of oceanic motions. Quantitative measurements are only to be had exceptionally. The motions of the sea's surface is in many cases known qualitatively from the drift of floating objects or of bottles thrown out for the purpose of investigation. Qualitative conclusions as to the motions in the deeper sheets can be drawn from the measurements of the salinity, this giving information as to the origin of the waters. Similar conclusions can be made also on the basis of the examination of the organisms contained in the water. But none of these observations are of the quantitative nature which can give rise to a closer kinematic analysis.

For this reason we can work out no example of a kinematic diagnosis of the sea-motions. But the principle of the methods to be employed in the case of the sea, as soon as serviceable material of observations is produced, will be sufficiently illustrated by the example worked out for the case of the atmosphere. We shall therefore only make occasional references to the sea.

The most important point to emphasize is the necessity of producing sufficient data of direct observations of the sea-motions from the surface as well as from all depths. Suitable instruments for doing it have already been invented.* It remains only to bring them into application on a sufficiently large scale and according to rational principles.

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CHAPTER III.

ELEMENTARY PRINCIPLES OF KINEMATICS OF CONTINUOUS MEDIA.

109. **Kinematics of a Continuous Medium.**—We have considered the observations from which we shall derive our diagnosis of atmospheric or hydrospheric states of motion. We shall then proceed to develop the general principles of kinematics which shall govern the diagnostic work.

In order to arrive at these general principles, we shall consider atmosphere, hydrosphere, and solid earth as a material system which fills space continuously. We shall neglect phenomena related to the molecular structure of these bodies, such as the diffusion of water-vapor through air or of salt through water. In the same manner we shall neglect every transfer of mass from one of these bodies to any other of them. Thus we shall set out of consideration the transfer of mass from the sea or from the moist ground to the air by the evaporation of water, and the return of these masses to the sea or to the porous ground in the form of rain. These processes will be of high importance in connection with the thermodynamics of atmosphere and hydrosphere. But from the pure kinematic point of view they will be insignificant, as they will give mass-transports which are small compared with those connected with the great air-motions or sea-motions.

It will therefore be sufficient for our present purpose to consider a material medium which fills space continuously. Density or specific volume may vary from particle to particle of the medium, even in discontinuous manner, as at the surface of separation between air and sea. The dynamic properties are not taken into consideration. The only condition to be observed is that of the material nature of the medium, involving the principle *that every moving particle shall have an invariable mass*, together with the supplementary condition *that the medium shall fill space continuously*.

To describe the instantaneous state of motion of this medium we shall use two vectors, velocity and specific momentum. The conditions of the material nature of the medium, and of its continuity in space, do not restrict the generality of the fields of these vectors. The methods of representing them geometrically will therefore be the methods of representing geometrically a vector-field of unlimited generality. From a formal point of view this chapter will therefore deal with the subject of the geometrical representation of vector-fields, and will thus contain results which we shall use later in connection with other vectors.

While the conditions of the material nature of the medium and of its continuity in space do not restrict the geometrical properties of the field of motion, they will

lead to a fundamental relation connecting these fields with that of mass. For as motion consists in the displacement of invariable masses having to fill space continuously, the knowledge of the present field of motion involves a certain knowledge regarding the future field of mass. Thus the two fundamental suppositions regarding the medium lead to an intrinsic relation of *prognostic* nature, in its mathematical form called the *equation of continuity*. In special cases time drops out, and the equation is reduced to a *diagnostic* one, submitting the fields of motion to certain restrictive conditions. In connection with the geometrical principles for representing the fields of motion, we shall therefore develop this prognostic equation and pay special attention to the cases when it is reduced to a diagnostic equation.

110. Vector-Lines, Vector-Surfaces, and Tensor-Surfaces.—In Statics we have considered the methods of representing geometrically certain special vectors, the ascendants or the gradients of a scalar quantity (sections 16, 17). The field of the scalar quantity gave a complete representation also of the vector derived from it. But in the general case a vector will, for its geometrical as well as for its analytical representation, require the use of three instead of only one scalar quantity.

In order to represent first the *direction* of a vector at every point of the field, we can draw a set of curves running tangentially to the direction of the vector. These lines are called *vector-lines*, or for a field of motion *lines of flow*. A set of curves in space is obtained by the intersection of two sets of surfaces. Each set of surfaces being the equiscalar surfaces of a certain scalar field, we see that the representation of the direction of a vector by vector-lines involves the use of two scalar fields.

The surfaces used to represent the vector lines may be chosen in an infinite number of ways; but they have the common property of being surfaces generated by vector-lines. Any surface generated in this way will be called a *vector-surface*, or for the field of motion a *surface of flow*.

The direction of the vector being thus given by two scalar fields, we can use a third for representing its numerical value or its *tensor*. An equiscalar surface of this third field will pass through all points where the vector has a certain constant numerical value. These surfaces may be called *tensor-surfaces*, or *surfaces of equal intensity*.

The vectors considered by us will have a uniquely determined direction at every point where it is different from zero. As intersections of vector-lines under finite angles would give two or more different directions for the vector in the point of intersection we conclude:

Vector-lines can intersect each other only at zero-points of the field.

Nothing prevents vector-lines from touching each other; for, having a common tangent, both lines indicate the same direction at the point of tangency.

111. Vector-Tubes and Surfaces of Equal Transport.—The two sets of vector-surfaces cutting each other along the vector lines will divide the field into a set of elementary tubes which have parallelogrammatic cross-sections. These may be called *vector-tubes*, or, for a field of motion, *tubes of flow*.

Cutting a vector-tube by any surface σ let $d\sigma$ denote the area of the section. A being the vector and A_n its component normal to the section, let us consider the product $A_n\ d\sigma$. This product does not depend upon the angle contained between the normal to the section and the axis of the tube; for as this angle varies, the area $d\sigma$ of the section and the vector-component A_n normal to it will vary in inverse proportion to each other, always giving a product equal to that of the area of the normal section into the tensor of the vector. We will call this product the *transport* through the section. The name is derived from the case to be examined more fully below, when the field represents motion. The bundle of tubes cutting through a finite surface σ divides this surface into elements $d\sigma$ and determines a certain transport $A_n\ d\sigma$ through each of them. Forming their sum, we get

$$(a) \qquad \textit{Transport through surface } \sigma = \int A_n d\sigma$$

The excess of transport leading out of a closed surface over that leading into it may be called the *outflow*, and the same quantity with the sign changed the *inflow*. The outflow is obtained by taking the integral (a) over the closed surface, counting the normal directed outward as positive. The inflow is obtained in the same way, counting the normal directed inward as positive.

Returning to an elementary vector-tube, let the section be moved from place to place along it. The transport will then, as a rule, be found to vary. Measuring its value from section to section in all tubes, we get numbers representing the *field of transport*. This field can be represented in the common way by drawing *surfaces of equal transport*.

The tubes of flow in connection with the surfaces of equal transport will give a representation of the vector field as complete as that given by the lines of flow in connection with the surfaces of equal intensity; for, being sufficiently narrow, the tubes will represent the direction of the vector equally well as the lines; and from the value of the transport we can come back to the numerical value of the vector dividing by the area of the cross-section of the tube.

Though the field of transport thus performs a similar service as the field of intensity for representing the numerical value of the vector, one important difference should be observed. The intensity-field is uniquely determined, while the field of transport has a definite sense only in connection with a given system of tubes. Choosing new surfaces for defining the tubes, we shall as a rule get tubes which have other cross-sections, and therefore lead to a new field of transport.

112. **Solenoidal Vector.**—A field may have the property that the outflow is zero out of every closed surface. The transport will then be the same through every section of one and the same tube. The surfaces of equal transport may then be left out as superfluous. It will be sufficient to know the constant of transport for each tube. It will in this case be found convenient to undertake the division of the field into tubes in such a way that each tube gets the same transport, in the simplest case unit transport. Choosing a unit of suitable magnitude, we can still get tubes sufficiently narrow for the purpose of representation. These narrow, in the limiting

case infinitely narrow, tubes are called *solenoids*, and every vector which can be represented completely by such tubes is called a *solenoidal vector*.

The solenoidal vector is simpler than the general vector inasmuch as it can be represented completely by two sets of surfaces, *i. e.*, by two scalar fields, while the general vector requires three. In other words, there is a dependency between the three components of the solenoidal vector. Using the *solenoidal condition*, *i. e.*, the condition expressing the fact that the outflow from every closed surface is zero,

(*a*) $$\int A_n d\sigma = 0$$

we can determine the third component of the vector, if we know the value of the two others at all points of the field.

113. Volume-Transport and Mass-Transport.—Passing to concrete fields of motion, we shall consider a tube of flow and a section of it having the area $d\sigma$. The particles situated at a certain time t on this section and having the velocity v, will an element of time dt later be situated on another section which is displaced the distance vdt along the tube. The normal distance between the sections will be $v_n\, dt$. The two sections and the walls of the tube determine an elementary parallelepipedon of volume $v_n\, dt\, d\sigma$, giving the elementary volume of the medium which during the time dt has passed the section $d\sigma$. Multiplying by the density ρ of the medium, we get the mass contained in this volume, *i. e.*, the mass which during the time dt has passed the section $d\sigma$. When we remember that the product of density into velocity gives the specific momentum V of the medium, we get as expression of this elementary mass $V_n\, dt\, d\sigma$. Dividing by dt we get the expressions $v_n\, d\sigma$ and $V_n\, d\sigma$ representing, according to our definition, the transport respectively in the field of velocity and in the field of specific momentum. We thus arrive at this result:

(A) *In the field of velocity the transport through a surface*

(*a*) $$\int v_n d\sigma$$

gives the volume of the medium passing the surface per unit time.

(B) *In the field of specific momentum the transport through a surface*

(*b*) $$\int V_n d\sigma$$

gives the mass of the medium passing the surface per unit time.

Taken over a closed surface the integral (*a*) will represent the volume and (*b*) the mass of the medium conveyed per unit time out through the closed surface.

Considering the transport as given in our m. t. s. units, and returning to the vectors, we arrive at these methods of measuring velocity and specific momentum, which may be useful to bear in mind in the subsequent practical work:

(C) *Velocity is measured by the number of cubic meters and specific momentum by the number of tons passing per second a square meter normal to the direction of the motion.*

114. Equation of Continuity.—The physical significance of the integral expressing transport in a field of motion being thus known, it will be easy to give in quantitative form the dependency of the future fields of mass upon the present field of motion.

Measuring the elementary volume conveyed out of a closed surface in an element of time dt, we evidently get the elementary increase of volume during the time dt of that mass which is momentarily contained in the closed surface. Reducing to unit time we get the velocity of expansion of this mass. Thus:

(A) *The integral of the normal component of velocity taken over a closed surface*

$$\int v_n d\sigma \tag{a}$$

is equal to the increase of volume per unit time of the mass momentarily contained in the surface.

Measuring on the other hand the elementary mass conveyed out of a closed surface in the element of time dt, we get the elementary decrease during this time of the mass stored within the surface. Reducing to unit time, we get:

(B) *The integral of the normal component of the specific momentum taken over a closed surface*

$$\int V_n d\sigma \tag{b}$$

is equal to the diminution per unit time of the mass contained in the surface.

The dependency of the future field of mass upon the present field of motion is expressed by these two theorems in two different ways, in the first case by the change of volume of moving masses, in the second by the change of mass within stationary volumes. We shall later write in explicit form the "equation of continuity," expressing in mathematical symbols the contents of any of these theorems. Provisionally it will be found more convenient to work directly with the physical facts as contained in the theorems (A) and (B).

115. Conditions Leading to Solenoidal Fields of Motion.—Every reference to variations in time of the field of mass will disappear from the theorem 114 (A) if the mass momentarily contained in the closed surface does not change its volume. In this case the field of velocity will fulfil the solenoidal condition

$$\int v_n d\sigma = 0 \tag{a}$$

In the same manner, the reference to future fields of mass will drop out of the theorem 114 (B) when the content of mass of every stationary volume is constant. Specific momentum will then fulfil the solenoidal condition

$$\int V_n d\sigma = 0 \tag{b}$$

We thus get the important results:

(A) *Velocity is a solenoidal vector if the moving medium is incompressible.*

(B) *Specific momentum is a solenoidal vector if the field of mass is stationary in space.*

If the medium be both incompressible and homogeneous, the moving masses will not change volume, and the mass-contents of every stationary volume will be invariable. We thus get the special case:

(C) *Both velocity and specific momentum will be solenoidal vectors if the moving medium be both homogeneous and incompressible.*

Without restricting the physical properties of the medium, we can apply theorem 114 (A) to the infinitely small volume contained between two parallel surfaces running at infinitely small distance from each other. Finite difference between the normal velocity-components at adjacent points on the two surfaces would in this case lead to finite expansion of an infinitely small volume. Thus the continuity would be broken. Therefore a finite difference between the normal components can not exist. This leads to the solenoidal surface-condition:

(D) *The normal component of velocity must have the same value on both sides of any surface in a material system filling space continuously.*

This solenoidal surface-condition must be fulfilled, for instance, at the surface of separation between atmosphere and hydrosphere. It applies only to velocity, not in general to specific momentum. Taking the case of mercury and water in contact with each other, the normal component of velocity will be the same on both sides of the surface; but that of specific momentum will be 13.6 times greater on the side of the mercury than on that of the water.

If the system is at rest on the one side of the surface, there will be no velocity-component normal to it on the other side; consequently the normal component of specific momentum will also be zero. Thus:

(E) *Velocity and specific momentum are directed tangentially to every resting boundary.*

This condition is to be applied to the motion of the air along the ground and of the water along the bottom of the sea.

116. Examples of Volume-Transport and Mass-Transport.—It will be useful here to take a few examples illustrating the difference between the conditions of solenoidal velocity-field and solenoidal field of specific momentum.

Let a tube be filled partly with water and partly with mercury, both fluids being considered incompressible. If the fluid column moves along the tube there will be equal volume-transport through a section in the water and through one in the mercury, say one cubic meter per second through each. The volume-outflow out of the closed surface formed by the walls of the tube and the two cross-sections will be zero, and the field of velocity will be solenoidal. But measuring the transport in tons, we find a transport of one ton per second through the upper and a transport of 13.6 tons per second through the lower section. The difference, 12.6 tons per second, gives the outflow of mass through the walls of the volume, and thus the decrease per unit time of the mass contained in the volume. We shall have outflow or inflow of mass according to the direction of the motion. For there will be a decrease of mass in the volume when water expels mercury, and an increase

when mercury expels water. The specific momentum will be solenoidal within each homogeneous part of the fluid column, but non-solenoidal at the surface of of discontinuity separating water and mercury. Instead of a discontinuous system like this, we could also have considered a fluid system with continuously varying density, for instance, a column of water with continuously varying salinity. Even in this case we would have a solenoidal velocity-field and non-solenoidal field of specific momentum, the solenoidal condition being violated by this vector not only at a certain surface of discontinuity, but at every point where density showed variations in space.

Let us now, on the other hand, consider motions in a compressible medium, atmospheric air. Setting aside the insignificant influence of humidity, we know that the density of the air depends upon temperature and pressure. Therefore, if the fields of temperature and of pressure are maintained stationary in space, the field of mass will also be stationary, and the specific momentum will be a solenoidal vector. Let us then consider a tube having its lower end near sea-level and its upper end in the region of cirrus. If one ton of air enters the tube per second at its lower end, one ton per second must leave it at its upper end. But measuring by volumes, we find that one ton of air has at sea-level a volume of about 1000 cubic meters, and at the height of cirrus a volume of about 3000 cubic meters. There is a volume-outflow from the closed volume limited by the walls and the cross-sections of the tube equal to 2000 cubic meters per second. This volume-outflow is equal to the velocity of expansion of the column of air which is contained in the tube. This expansion is due to the motion up toward lower pressures. Reversing the direction of the motion, we get a corresponding inflow, equal to the contraction per second which the column of air will have in virtue of its descending motion.

117. The Fields of Motion in Atmosphere and Hydrosphere.—We can now take up the discussion of the chances of arriving at a satisfactory diagnosis of atmospheric or hydrospheric motions. The great incompleteness of the observations of air-motions is that they give only the horizontal components, and no information on the vertical components. The same has also hitherto been the case with all observations of sea-motions. The conditions for a satisfactory diagnosis will then be that we should be able to derive the unknown vertical components from the observed horizontal components. This will be possible if the motions can be considered solenoidal, and the question will be if we can suppose this to be the case with sufficient approximation for the purpose of the kinematic diagnosis.

In the case of the hydrosphere there is no doubt. We can put out of consideration both the slight compressibility of the sea-water and the slight changes in the field of mass following local changes of temperature, salinity, and pressure. Doing so, we find that both the field of velocity and the field of specific momentum will fulfil the solenoidal condition. Using this condition for deriving the not-observed vertical components from the observed horizontal ones, we shall obtain an accuracy depending entirely upon that of the observations; for the errors introduced by neglecting compressibility and heterogeneity will be insignificant.

In the case of the atmosphere we have seen already that the changes of volume of the moving masses of air are too great to allow us to consider the field of velocity solenoidal. But the field of mass is not very far from being stationary, the changes in this field being caused exclusively by the gradual changes in the fields of temperature and of pressure; we may therefore try to derive the vertical motions, supposing the field of specific momentum to be solenoidal in the first approximation.

In order to see the errors which can then arise, we can consider a cylinder going from the ground up to a certain height in the atmosphere. Calculating the vertical motion through a horizontal section at the top of the cylinder, we set the transport of mass *up* through this section equal to the transport of mass *in* through the walls of the cylinder. The vertical motion thus found will be erroneous, inasmuch as the temperature or pressure within the cylinder is changing. To find the error we shall estimate the additional vertical motion produced by the local changes of temperature and pressure.

First let there be an increase of temperature within the cylinder of 1° C. per hour, *i. e.*, of $\frac{1}{3600}$° C. per second. This will give a cubic meter of air the velocity of expansion of $\frac{1}{273}\cdot\frac{1}{3600}$, or less than one-millionth of a cubic meter per second. The corresponding linear velocity of expansion of the air in the cylinder will be less than one micron per meter in the second. There will thus arise a vertical velocity not exceeding 1 mm. per second at the height of 1000 meters, and not exceeding 1 cm. per second at the height of 10,000 meters. We can only as an exception expect to get changes of temperature greater in average than a few degrees centigrade per hour for columns of air of this height. Therefore, neglecting the local change of temperature, we shall get errors in the vertical velocities not exceeding a few millimeters per second at the height of 1000 meters, and a few centimeters per second at the height of 10,000 meters.

For the corresponding influence of local change of pressure, we can suppose temperature to be constant. For the column of air contained in the cylinder we have then $pK = \text{const}$, p being the average pressure in the cylinder and K its volume. As only the height z of the cylinder is variable, we can write $pz = \text{const}$. Differentiating with respect to time and solving with respect to $\frac{dz}{dt}$, we get

$$\frac{dz}{dt} = -\frac{z}{p}\frac{dp}{dt}$$

Supposing the change of pressure $\frac{dp}{dt}$ to be one m-bar per hour, *i. e.*, $\frac{1}{3600}$ m-bar per second, setting the height of the cylinder equal to 1000 meters, and the average pressure between sea-level and this height equal to 900 m-bars, we get the vertical velocity $\frac{dz}{dt}$ smaller than a third of a millimeter per second. Setting the height of the cylinder equal to 10,000 meters and the average pressure between this level and sea-level equal to 600 m-bars, we get the vertical velocity due to the variation of the pressure smaller than half a centimeter per second. Thus in both cases the

change of pressure of one m-bar per hour will have smaller effect than the change of temperature of one degree centigrade per hour. Now the change of pressure of a few m-bars per hour for columns of air of this length will have about the same degree of probability as the change of temperature of a corresponding number of degrees. Thus we have an equal right to neglect the influence of local pressure-changes as of local temperature-changes.

When we determine vertical velocities in the atmosphere by the condition of the solenoidal nature of specific momentum, we may thus get errors amounting to a few millimeters per second at the height of 1000 meters and of a few centimeters per second at the height of cirrus. As the errors due to the uncertainty and the incompleteness of the observations of the horizontal velocities will be much greater, these errors must be considered as insignificant.

We can therefore set down as fundamental principles to be used in the diagnostic work regarding the fields of motion in atmosphere and hydrosphere:

(A) *In hydrosphere both velocity and specific momentum fulfil the solenoidal condition.*

(B) *In atmosphere specific momentum fulfils the solenoidal condition.*

Finally we have, independent of every approximation:

(C) *At every surface velocity fulfils the solenoidal surface-condition.*

As a special case of this condition we have

(D) *Both velocity and specific momentum are tangential to every resting boundary.*

CHAPTER IV.

EXAMPLES OF SOLENOIDAL FIELDS AND THEIR REPRESENTATION BY PLANE DRAWINGS.

118. Two-Dimensional Representations of Three-Dimensional Vector-Fields.—Before passing to practical applications, it will be useful to consider a few simple examples of solenoidal fields and to illustrate different methods of representing them by plane drawings.

In order to see the character of two-dimensional drawings representing any three-dimensional field, let us consider a surface cutting through the field in space. At every point of the surface the vector has a certain direction and intensity. For the representation it will be convenient to consider separately two projections of the vector, that on the normal to the surface, and that on the plane tangential to the surface. *The normal component* can be represented simply by curves for equal numerical values. No representation of the direction is required. The field of this component has lost the character of a vector-field and has completely obtained that of a two-dimensional scalar field.

The tangential component, on the other hand, will represent a true *two-dimensional vector-field*. The methods of representing it geometrically will be special cases of the methods for representing vectors in space (section 110). Precisely as in space, the direction can be represented by *vector-lines*. But instead of surfaces we shall get *curves of equal intensity*. It should be observed that these curves of equal intensity will not, as a rule, be the intersections of the given surface with the surfaces of equal intensity in space. This will be the case only if the given surface happens to be a vector-surface; for then the normal vector will disappear and the vector of the two-dimensional field will be identical with that of the three-dimensional field in space.

A set of two-dimensional drawings representing a three-dimensional vector-field in space can therefore be obtained in the following way: We choose a set of surfaces cutting through the field. The vector defines at each of them a two-dimensional vector-field and a two-dimensional scalar field. The first can be represented by two sets of curves, viz, the vector-lines and curves of equal intensity for the tangential component; and the second by one set of curves, viz, curves for equal values of the normal component.

We shall then have to direct our attention to the two-dimensional vector-fields contained in a surface and to the correlated scalar fields representing a vector-component normal to the surface.

119. General Remarks on the Two-Dimensional Vector-Field.—For the same reasons which we have for vector-lines in space, we get:

> *Vector-lines of the two-dimensional field can intersect each other under finite angles only at points where the two-dimensional vector is zero, i. e., at points where the corresponding vector in space is either zero or normal to the surface containing the two-dimensional field.*

Instead of vector-tubes we shall in the two-dimensional field get *vector-bands* bordered by vector-lines. Transport must be referred to lines instead of to surfaces. A_n being the component of the vector normal to the curve s, we get

(a) $$\textit{transport through curve } s = \int A_n ds$$

Instead of surfaces we get *curves* of equal transport. The solenoidal condition is expressed by

(b) $$\int A_n ds = 0$$

the integral being extended over a closed curve. When condition (b) is fulfilled, the curves of equal transport can be left out, and the field be represented by bands of equal transport, most conveniently of unit transport. If unit bands be used, the numerical value of the vector is given by the reciprocal of the number expressing the breadth of the band.

If a unit band gets infinitely narrow, the solenoidal vector will be infinite. Excluding infinite values, we get this important result:

In the two-dimensional solenoidal field the lines of flow can not touch each other.

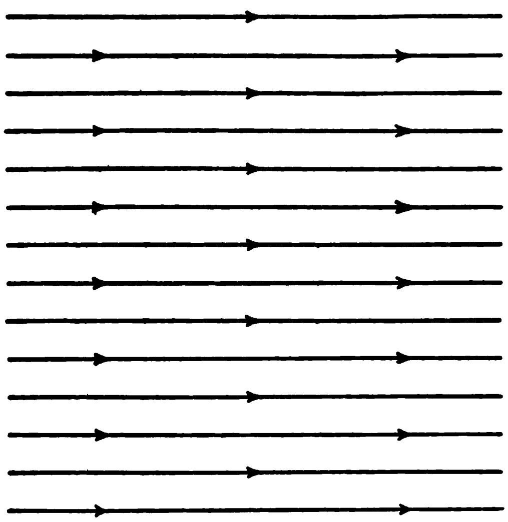


FIG. 35.—Translation-field.

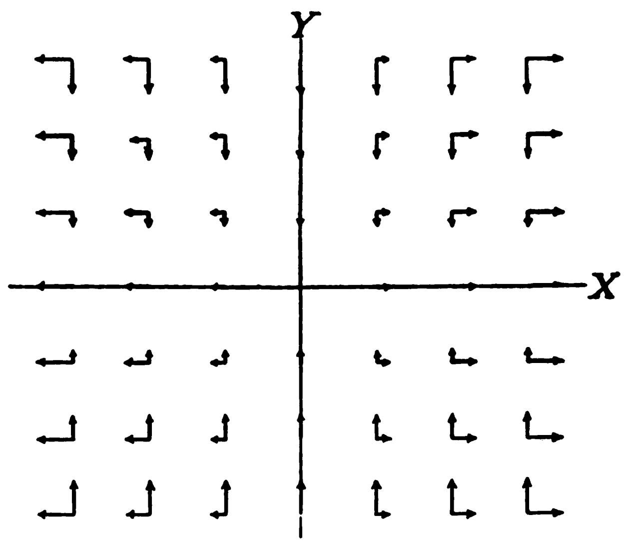


FIG. 36.—Vector-components of a plane deformation-field.

120. Examples of Two-Dimensional Solenoidal Fields.—We shall consider first the case that the two-dimensional field is solenoidal. Let the surface containing the field be plane. The simplest field will be that of a vector having the same direction and the same intensity at all points of the plane. If the vector is velocity the field will represent simple motion of translation. The field evidently fulfils the solenoidal condition. It can be represented geometrically by a set of parallel and equidistant vector-lines (fig. 35).

Let us next consider a field where the component A_x parallel to the axis x is proportional to x, and the component A_y parallel to the axis y is proportional to y:

(a) $$A_x = ax \qquad A_y = by$$

The line-integral of the normal component of the vector is easily found for any closed curve having the form of a rectangle with sides parallel to the axes of coordinates. If two of the sides are the coordinate axes, and the two others the lines $x=x$ and $y=y$, the line-integral taken over the closed curve will be $A_x y + A_y x$. Substituting the values (a) of the components, we get the line-integral equal to $axy+bxy$, and the solenoidal condition is seen to be fulfilled if $b = -a$. Thus the formulæ will be

$$(b) \qquad A_x = ax \qquad A_y = -ay$$

Fig. 36 represents the components of this solenoidal vector. If the vector represents velocity, the motion given by fig. 36 will be the simplest typical fluid motion producing a *deformation* of the fluid masses without change of volume.

A vector-line is determined by the condition that the projections dx and dy of its line-element are proportional to the vector-components A_x and A_y. It is therefore given by the differential equation

$$(c) \qquad \frac{dx}{A_x} = \frac{dy}{A_y}$$

Substituting the values of A_x and A_y according to (b), and integrating, we find

$$(d) \qquad xy = \text{const.}$$

i. e., the vector-lines are equilateral hyperbolæ, having the axes of coordinates as asymptotes (fig. 37). These axes themselves belong to the system of lines of flow,

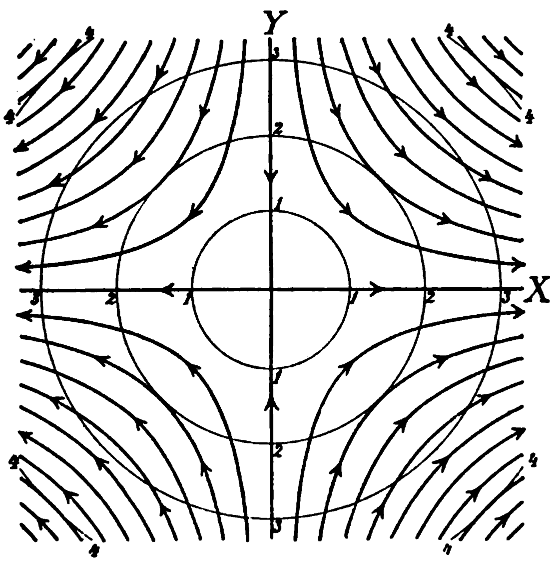


FIG. 37.—Hyperbolic lines of flow and circular curves of equal intensity 1, 2, 3, . . . of a plane deformation-field.

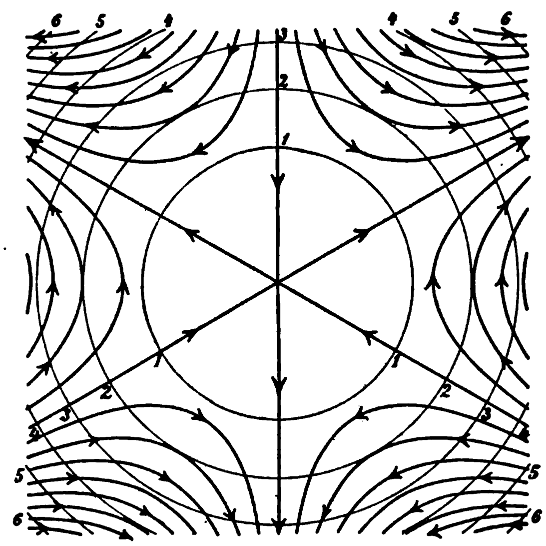


FIG. 38.—Neutral point of higher order.

and cut each other at the *neutral* point of the field where $A_x = A_y = 0$. The intensity A of the vector is

$$A = \sqrt{A_x^2 + A_y^2} = a\sqrt{x^2+y^2} = ar$$

Thus the curves for equal intensity are circles $r=$const. around the neutral point. Through equal lengths of a line parallel to one of the axes there will go equal transport. Drawing the hyperbolæ through equidistant points on such a line as made in

fig. 37, we get bands of equal transport and can leave out the curves of equal intensity. If this field represents a field of motion, it gives the picture of two currents flowing against each other, bending off against each other, and canceling at the neutral point.

Neutral points of a more complex nature, where three or more currents cancel simultaneously, may also be conceived (fig. 38).

121. Graphical Addition of Two-Dimensional Solenoidal Fields.—The investigation of the two-dimensional solenoidal vectors is much assisted by a construction allowing us to pass from the representations of the fields of two such vectors to that of their vector-sum.

Let the two given fields be represented by the two sets of thin lines of fig. 39. These lines divide the plane into a set of parallelograms. Every diagonal in any one of the parallelograms represents a section simultaneously of two unit bands, viz, of one belonging to the first and of one belonging to the second of the given fields. It is further seen that through one diagonal in a parallelogram goes the sum of the transports in two unit bands, *i. e.*, the transport 2, while through the other goes the difference, *i. e.*, the transport zero. Drawing the diagonal curves formed by the

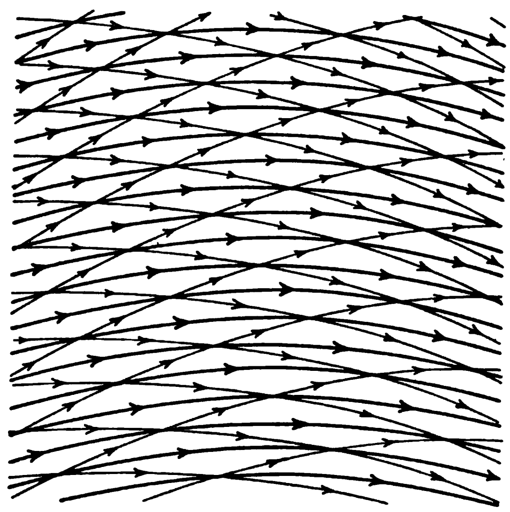


Fig. 39.—Graphical addition of two-dimensional solenoidal fields.

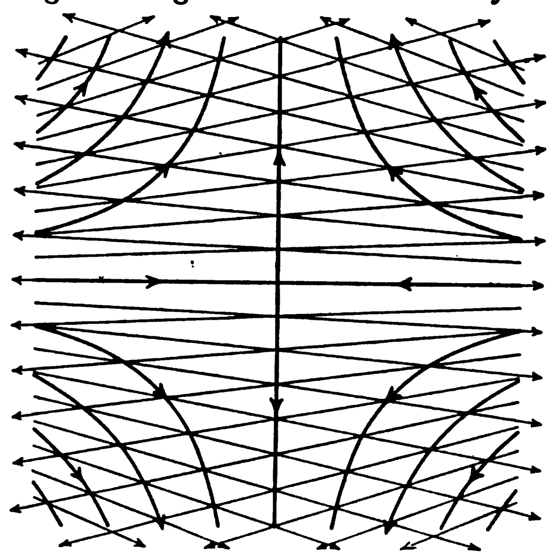


Fig. 40.—Addition of oppositely directed divergent fields.

latter set of diagonals, we evidently get lines of flow of the field due to the coexistence of the two given fields. These lines are drawn heavy in fig. 39. Further, it is seen that the bands separating these lines are unit bands. For two of them correspond to each diagonal through which we found the transport to be equal to 2.

As an application of the construction, fig. 40 shows how a deformation-field with neutral point and hyperbolic vector-lines is produced by the coexistence of two oppositely directed fields with straight, slightly divergent vector-lines.

Figure 41 shows the effect of adding the field with parallel and equidistant straight vector-lines to that with the hyperbolic vector-lines. As is seen, the result is simply a displacement of the latter field, the neutral point turning up where the two fields cancel.

122. Solenoidal Field in Space with Neutral Point.—It will be useful to show the simplest case of a solenoidal field in space having a neutral point. Corresponding to the two-dimensional field of section 120, we shall then consider a field with the rectangular components

(*a*) $$A_x = ax \qquad A_y = by \qquad A_z = cz$$

The integral of the normal component of the vector is easily formed for a surface of parallelepipedic form having sides parallel to the coordinate planes. The solenoidal condition is seen to be fulfilled if

$$a+b+c=0$$

In order to simplify we shall further set $b = a$, which gives $c = -2a$. We then have the field

(*b*) $$A_x = ax \qquad A_y = ay \qquad A_z = -2az$$

Composing the components A_x and A_y we get a resultant contained in a plane passing through the axis of z. Calling r the distance of any point in this plane from the axis of z and R the resultant of A_x and A_y, we get instead of the two first equations $R = ar$. The field will then be completely given by the two components

(*c*) $$R = ar \qquad A_z = -2az$$

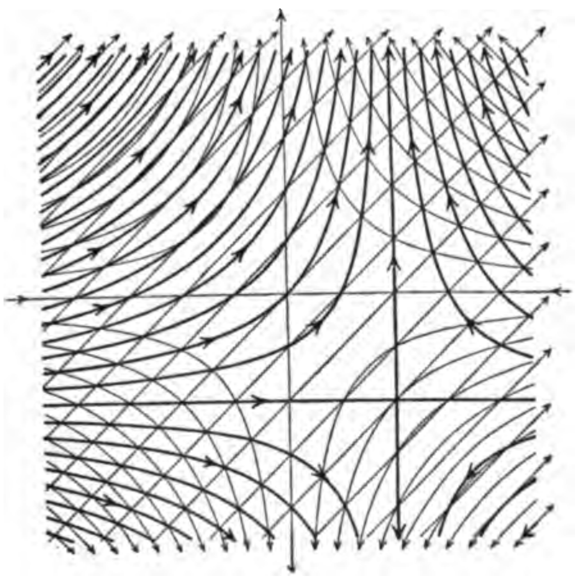


FIG. 41.—Addition of translation-field and deformation-field.

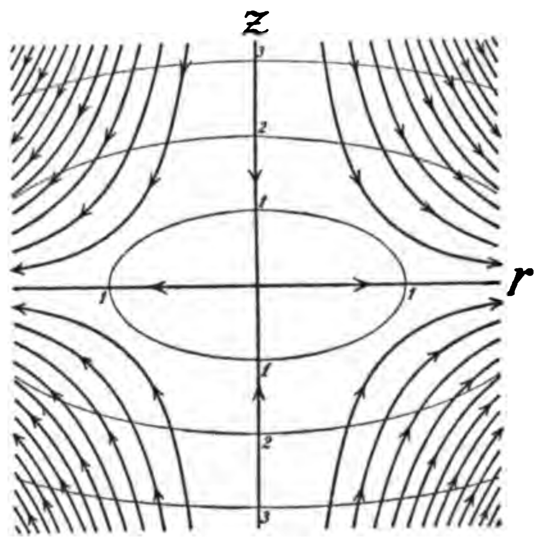


FIG. 42.—Lines of flow and curves of equal intensity, 1, 2, 3, of a symmetrical deformation-field in space.

The field is thus symmetrical around the axis of z, and the vector is contained in the meridian planes passing through this line. Substituting the values of R and z in the differential equation

$$\frac{dr}{R} = \frac{dz}{A_z}$$

and integrating, we get the equation of the vector-lines

(*d*) $$r^2 z = \text{const.}$$

They are a kind of asymmetric hyperbolæ having the axes of r and z for asymptotes, but converging more rapidly toward the first of these axes than toward the second (fig. 42). The axes are themselves vector-lines cutting each other at the neutral point of the field.

The vector is seen to have the constant numerical value A on the curve

$$A^2 = R^2 + A_z^2 = a^2r^2 + 4a^2z^2$$

which is an ellipse of half-axes $\frac{A}{a}$ and $\frac{1}{2}\frac{A}{a}$. These ellipses are drawn in fig. 42 for the values 1, 2, 3, of A.

We can now get a complete picture of the field. The meridian planes passing through the axis of z form one set of surfaces of flow. The other set is generated by the lines of flow of fig. 42, when this figure rotates around the axis of z. Simultaneously the other curves of this figure will generate the surfaces of equal intensity. We get thus the complete representation of the field by three sets of surfaces: two sets of surfaces of flow cutting each other along the lines of flow in space, and one set of surfaces representing equal scalar values of the vector.

As the field is solenoidal, a representation can also be obtained where the last set of surfaces is left out. A_z is constant in a plane z = const. Thus there goes equal transport through equal areas of this plane. A division of this plane into equal areas is obtained if the radial lines defining the meridian planes are drawn with equal angular intervals and the circles defining the other surfaces of flow are drawn with radii proportional to the numbers $\sqrt{1}$, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{4}$, ... These intervals have been chosen already for the meridian curves of fig. 42, which represent these surfaces of flow. Thus the intersection of these surfaces with meridian planes which have constant angular distance from each other will produce tubes of equal transport representing the field completely. The surfaces of equal intensity may then be left out.

As atmosphere and hydrosphere have a limited extent in vertical direction but an enormous extent in horizontal direction, the best representations of fields of motion in these media will be obtained by charts in horizontal projection. It will be useful to consider different types of charts representing a simple field of motion, as that which we have just examined. Fig. 43 gives four different types of such charts.

(A) In fig. 43 A, the full-drawn concentric circles are contour-lines representing the topography of one of the surfaces of flow, namely, that of which a profile-curve is drawn at the top of the figure. The radial lines represent the lines of flow on this surface. Their vertical course is given directly by the topography of the surface. Finally the stippled circles are curves for the equal intensity of the vector. Evidently a set of charts of this kind each containing three sets of curves, contour-lines, lines of flow, and curves of equal intensity, will give a complete representation of the field.

(B) A varied method of representation, derived from the solenoidal property, is given in fig. 43 B. The contour-lines giving the topography of a surface of flow are retained and the lines of flow on it are drawn as before. But these lines are

supposed to represent the projections of vertical walls separating from each other a set of tubes of flow. A third set of lines is then drawn, representing the height of these tubes. The curves for the equal height of the tubes will be a new set of con-

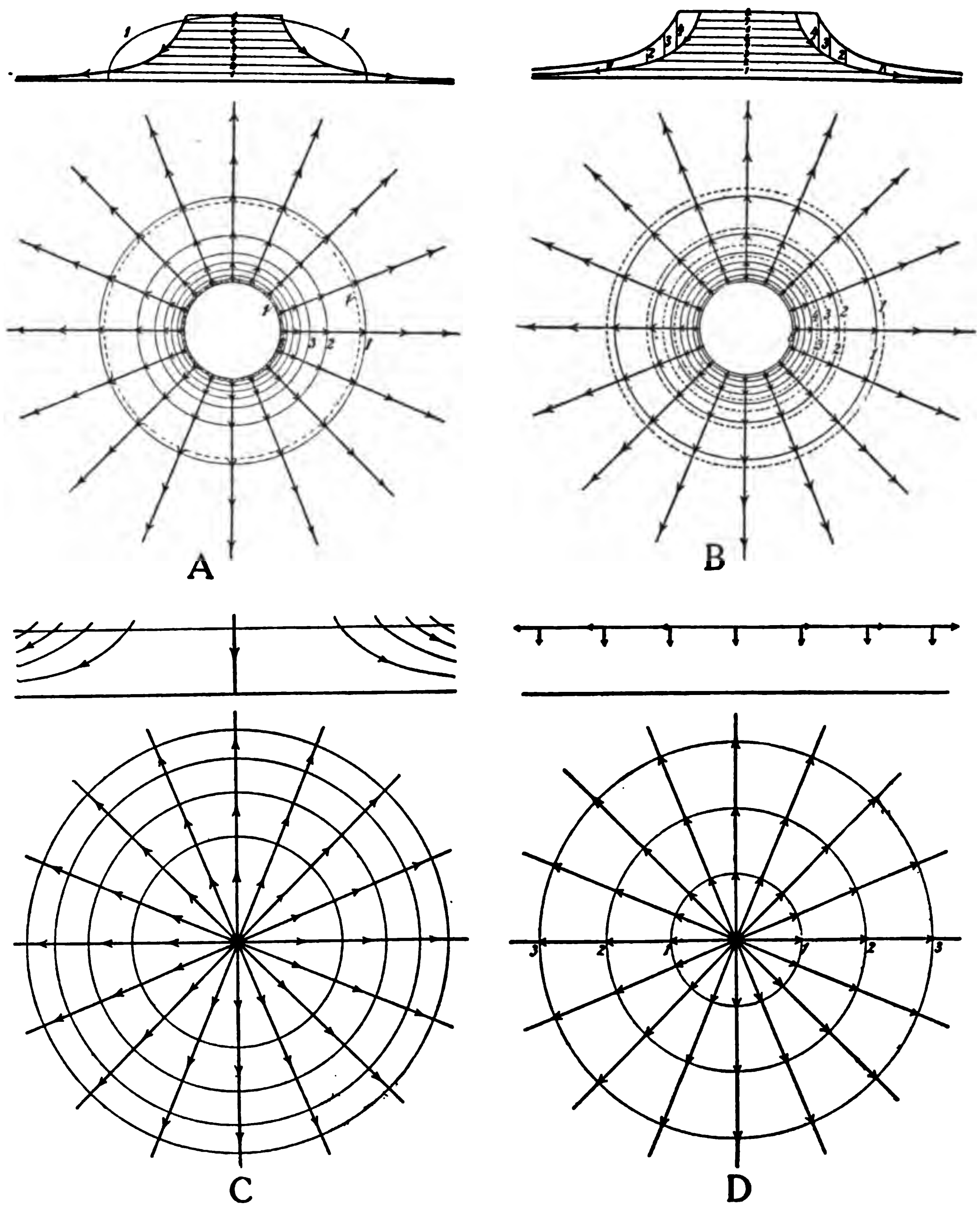


FIG. 43.—Field with singular point in space represented by different charts in horizontal projection.

A. Surface of flow represented by contour-lines (circles) and containing lines of flow (radii) and curves of equal intensity (stippled circles).
B. Tnbes of flow represented by contour-lines for absolute and for relative topography (full and stippled circles) and lines of flow (radii).
C. Horizontal section through the system of tubes of flow.
D. Two-dimensional field in a horizontal plane represented by lines of flow (radii) and curves of equal intensity (circles). Normal component constant and not represented.

tour-lines, giving the topography of a second surface of flow relatively to the first. The stippled circles of fig. 43 B are these contour-lines. A set of charts of this kind, each containing three sets of curves, lines of flow, curves of absolute and curves of relative topography, will also give a complete representation of the field.

Instead of using surfaces of flow, as in the cases (A) and (B), we can use arbitrary surfaces cutting through the field. We can then simplify by choosing surfaces of simple configuration, instead of the surfaces of flow, which as a rule will not be simple. But in return we must give special representations of the component fields tangential to and normal to the surface. In the case before us it will be easiest to cut the field by horizontal planes z = const. As above, we shall then get two different representations according as we make explicit use or not of the solenoidal property of the field. We shall then arrive at the following two representations, (C) and (D):

(C) Let us imagine the field in space to be given by tubes of equal transport *i. e.*, by the meridian planes and the surfaces of revolution which form the walls of these tubes. The two sets of surfaces will cut the horizontal plane in two sets of curves, the radii and the circles of fig. 43 C. These curves divide the plane into areas which are sections of the unit tubes, and thus areas of equal transport normal to the plane. While these areas represent the normal component-field, the radial lines of flow represent the tangential field. Evidently the field in space can be represented completely by a set of charts of this description.

(D) Instead of using the solenoidal property of the field, we can draw the vector-lines and the curves of equal intensity which represent the tangential field contained in the plane z = const, and the curves of equal intensity which represent the normal field, as developed in section 118. In the case before us the vector tangential to any of the planes z = const. is $R = ar$. It has radial lines of flow and curves of equal intensity which are concentric circles with radii increasing in arithmetical series (fig. 43 D). As in the case before us the normal component $A_z = -2az$ is independent of the coordinates x and y, no curves for representing the normal field are required. Only the constant value of the component will have to be noted for each plane.

123. Solenoidal Field in Space with Asymptotic Line.—As another example of a solenoidal field in space, we shall consider that defined by the rectangular components

(*a*) $$A_x = ax \qquad A_y = b \qquad A_z = -az$$

It consists of two partial fields which we have examined already (section 120), the field of the constant vector A_y and the field of the linear vectors A_x and A_z defining a two-dimensional deformation-field in planes parallel to the xz-plane. Each of these partial fields being solenoidal, that produced by their co-existence will also be solenoidal.

The vector-lines of the field thus produced will be represented by the differential equations

(*b*) $$\frac{dx}{A_x} = \frac{dy}{A_y} = \frac{dz}{A_z}$$

or, substituting the values of the components,

(c)
$$\frac{dx}{ax} = \frac{dy}{b} = -\frac{dz}{az}$$

Integrating each of the three equations contained in this system, we get

(d)
$$x = e^{\frac{a}{b}(y+c_1)} \qquad xz = c_2 \qquad z = e^{-\frac{a}{b}(y+c_3)}$$

The surfaces for equal scalar values A of the vector are given by the equation

(e)
$$A^2 = A_x^2 + A_y^2 + A_z^2 = b^2 + a^2(x^2 + z^2)$$

representing for every constant value of A a circular cylinder around the axis of y.

The second equation (*d*) shows that the lines of flow in space project themselves as equilateral hyperbolæ on the plane of xz. As the cylindrical surfaces of equal intensity cut this same plane along concentric circles, we get in this plane a figure precisely similar to that of fig. 37. The third equation (*d*) shows that the lines of flow in space project themselves on the yz-plane as exponential curves converging asymptotically toward positive y. The first equation (*d*) shows in the same manner that the lines of flow in space project themselves on the xy-plane as exponential curves diverging out asymptotically from negative y (fig. 44).

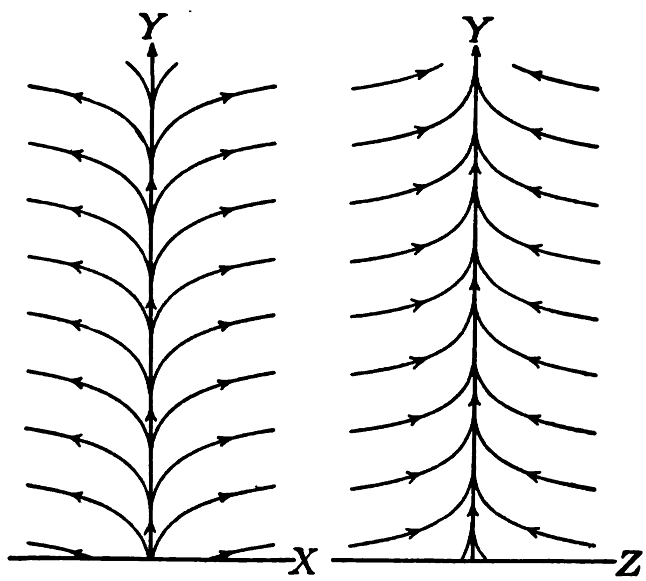


FIG. 44.—Lines of flow in the xy-plane diverging from, and in the yz-plane converging to the axis of y, which is a singular line of flow.

As the planes of xy and yz are themselves surfaces of flow, fig. 44 represents directly the lines of flow contained in these planes. The axis of y is itself a singular line of flow, and toward this line an infinity of lines of flow converge in asymptotically in the vertical plane and diverge out asymptotically in the horizontal plane.

In order to get a more complete view of the field, we can use the different representations by charts in horizontal projection.

(A) Fig. 45 A gives the topographical representation of two surfaces of flow which cut the xz-plane along two equilateral hyperbolæ. The course in space of

the lines of flow contained in these hyperbolic surfaces and projecting themselves on the horizontal plane as exponential curves is thus easily conceived. Adding the lines of equal intensity (stippled straight lines), we get a complete representation of the field contained in these hyperbolic surfaces.

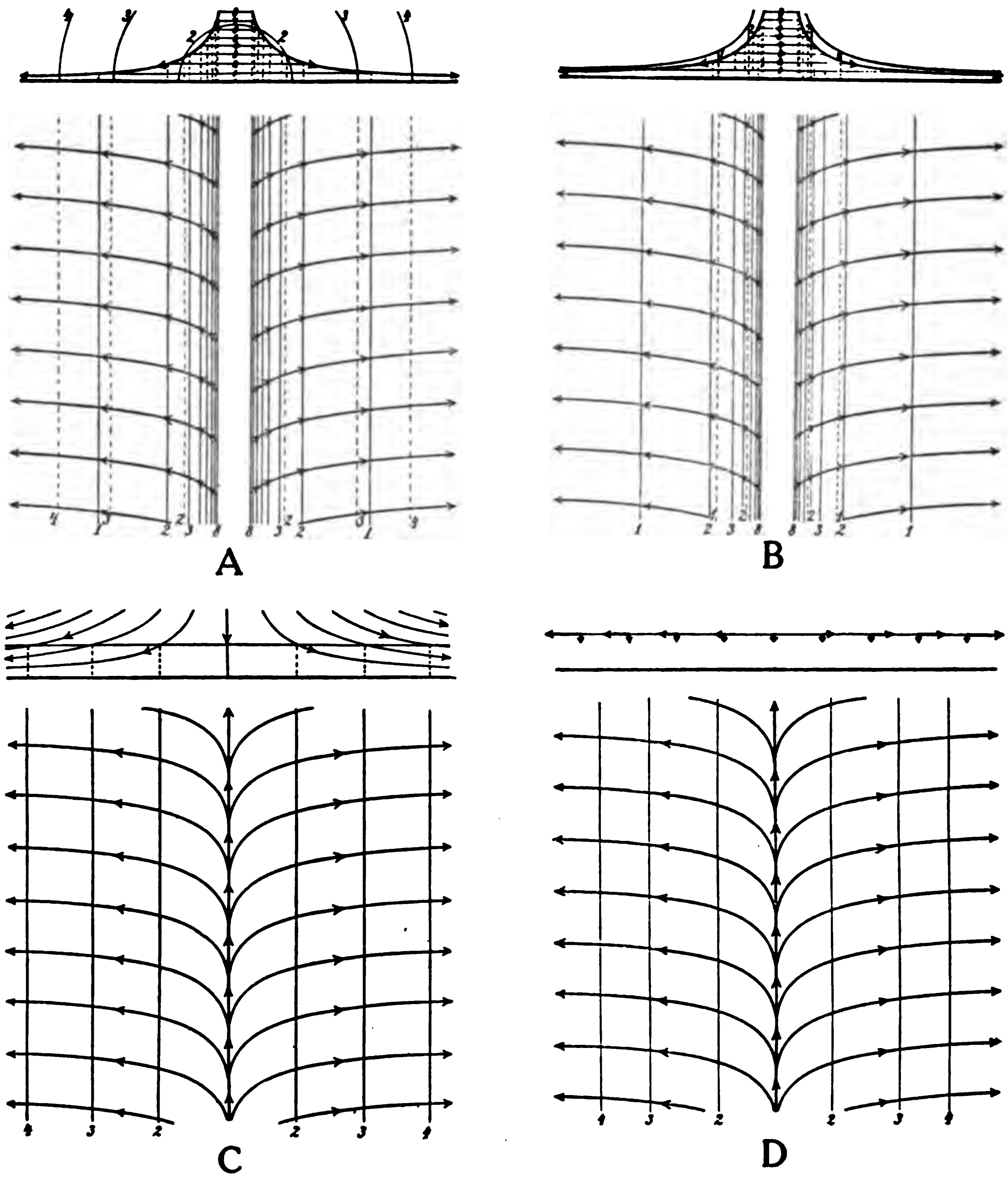


FIG. 45.—Field with asymptotic line in space, represented by different charts in horizontal projection.

A. Surface of flow represented by contour-lines (straight) and containing lines of flow (exponential curves) and curves of equal intensity (stippled straight lines).
B. Tubes of flow represented by contour-lines for absolute and for relative topography (full-drawn and stippled straight lines), and lines of flow (exponential curves).
C. Horizontal section through the system of tubes of flow.
D. Two-dimensional field in horizontal plane represented by lines of flow (exponential curves) and curves of equal intensity (straight lines).

(*B*) Leaving out the lines of equal intensity, and introducing in their place contour-lines giving the relative topography of a second surface of flow over the first, we get the solenoidal representation of the field contained between the two surfaces of flow (fig. 45 B).

(C) Fig. 45 C gives the horizontal section through the system of unit-tubes. The diagram shows the horizontal projection of the tubes and represents the vertical motion by a division of the horizontal plane into areas of equal transport normal to this plane.

(D) Fig. 45 D gives the lines of flow and the curves of equal intensity for the two-dimensional field contained in a horizontal plane. As in the example in section 122 (*D*), the vertical component $A_z = -az$ is independent of x and y and does not therefore require any special representation. But the principle of representing a variable normal component by drawing equiscalar curves is evident at once.

124. Charts Representing Fields of Motion in Atmosphere and Hydrosphere.— Referring to simple examples, we have given four different types of charts for representing fields of motion in space. Each type can be used practically in the case of atmospheric or hydrospheric motions, and we shall later indicate the methods of arriving at each of them. For the purpose of representation each type will have special advantages and special disadvantages. But it would lead too far to develop and exemplify them all in full detail. We shall therefore choose one of the methods as the principal one, namely the method D, *i.e.*, we shall choose arbitrary surfaces cutting through the field, and consider separately the two-dimensional vector-field contained in the surface and the scalar field representing the normal component of the vector.

As surfaces cutting through the field, we shall use level surfaces, isobaric surfaces, or for more limited purposes surfaces running parallel to the ground. In order to reduce as much as possible the number of drawings, we shall compose the two-dimensional vector-fields for a series of surfaces. In this manner we shall get two-dimensional vector-fields representing the *average tangential motion* within sheets of a certain thickness, level sheets, isobaric sheets, or sheets parallel to the ground. We have already made the introductory steps for the determination of such two-dimensional vector-fields from the observations (Chapter II).

These two-dimensional vector-fields being found as the direct result of the observations, we shall afterwards use the solenoidal condition for deriving the corresponding scalar fields representing the normal component of motion. It will be most convenient to determine them for the surfaces separating from each other the sheets for which the two-dimensional vector-fields have been drawn.

The methods for deriving the two-dimensional vector-fields from the observations will be considered in Chapters V–VII. Then Chapters VIII and IX will give from general points of view the graphical methods of performing mathematical operations to be used in the subsequent work. These methods being developed, we shall apply them in Chapters X and XI to complete the kinematic diagnosis by deriving the scalar fields which represent the normal component of the motion.

CHAPTER V.

DIRECT DRAWING OF THE LINES OF FLOW AND THE CURVES OF EQUAL INTENSITY FOR THE TWO-DIMENSIONAL VECTOR-FIELDS.

125. Continuous Representation of the Two-Dimensional Vector-Fields.—Passing to the practical diagnostic work, our first problem will be this: From the observations of motion (local values or averages for certain sheets) to draw the lines of flow and the curves of equal intensity for the corresponding two-dimensional field. Drawing these curves we shall get a continuous representation of this field instead of the discontinuous representation given by the observations themselves.*

Our solutions of concrete problems of this kind are given on plates XXXII, XXXVIII, LV, and LVII B to LX B. The lines of flow are represented by heavy curves provided with arrow-heads, the curves of equal intensity by thinner curves.

As such continuous representations of the two-dimensional fields are to form the basis for every further step in kinematic diagnosis or prognosis, we can not discuss too carefully the methods for drawing them as correctly as possible. Referring to the mentioned plates as examples, we shall take up this discussion, which will occupy us in this as well as in the two following chapters.

126. Equiscalar Curves in the Field of Single-Valued Scalar Quantities.—The numbers representing the numerical value of the vectors velocity or specific momentum define a scalar field having the same geometrical features as the well-known fields of other scalars, like pressure or temperature. The method of drawing the curves of equal intensity of a vector is therefore precisely the same as that of drawing isothermal or isobaric curves; but as the curves in the case before us will have an irregular course, the drawing will require a good deal of care.

Equiscalar curves are never drawn exclusively by the use of the numbers representing the observations. Otherwise an infinite number of observations would be required for the determination of their course. The intrinsic properties of the scalar quantity are also taken into consideration. The main property used in drawing the common synoptical charts is this, that the scalar is *single-valued*. As it can never have two different values in one point, *two different curves, representing different values of the scalar, can never intersect each other*. This property gives to the field of the single-valued scalar features which are totally different from those of the multiple-valued scalar, which we shall have to consider later.

*That charts of this character have not yet been used in practical meteorology, must be on account of their apparent complexity. The only charts containing lines of flow of atmospheric motions which we have been able to find in literature have been drawn by René de Saussure (Archives des Sciences Physiques et Naturelles, Quatrième Période, T. 5, p. 497, Genève, 1898) and by Jean Bertrand (Bulletin de la Société belge d'Astronomie et de Meteorologie, 1905, No. 7 and 8; see also Physikalische Zeitschrift, 1905, p. 853).

The property of never intersecting each other very much limits the course of the curves, and makes it possible to draw them as soon as the values of the scalar are known in a relatively small number of points. But the course is never completely determined by a limited number of observations. There will always be a certain limited freedom in the way of drawing each curve. But as the number of observations is increased this freedom will be reduced, and finally the course of the curve will be perfectly determined from a practical point of view, *i. e.*, with a certain finite degree of precision.

The curves will obtain their characteristic features by the situation of the points where the scalar has its extreme values. At the maximum points and the minimum points the equiscalar curve will be reduced to a point. These points are surrounded by closed equiscalar curves. Between the maximum and minimum points there will be maximum-minimum points. In each a certain singular equiscalar curve cuts itself. The two branches of this singular curve divide the field in the neighbor-

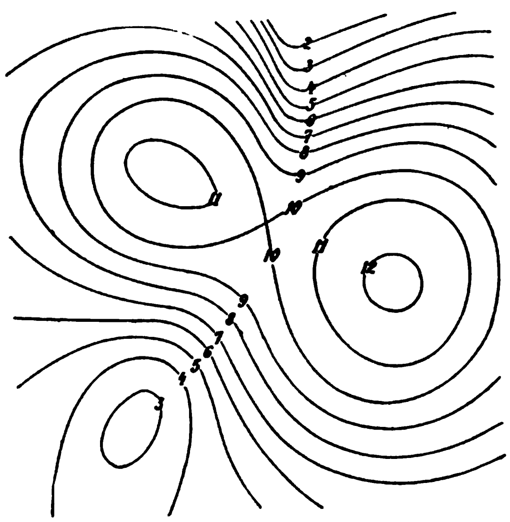


Fig. 46.—Maximum points, minimum points, and a maximum-minimum point of a scalar field.

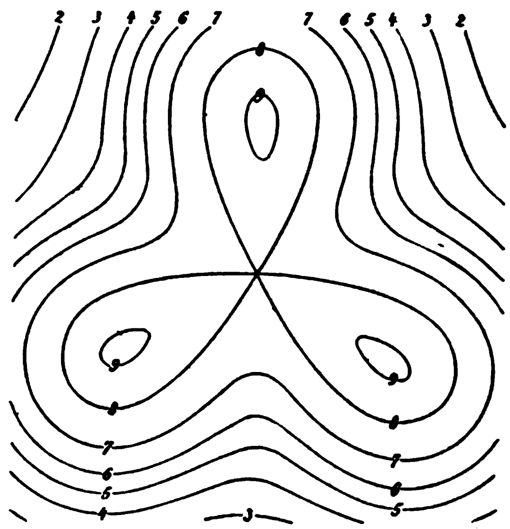


Fig. 47.—Maximum-minimum point of higher order.

hood of the maximum-minimum point into four angular areas. In two of them the scalar has greater and in two of them smaller values than in the point of intersection (fig. 46). More complex maximum-minimum points may also be mentioned, though they will rarely be met with in practice. Thus three branches of the singular equiscalar curves may cut each other in this point, dividing the surrounding field into six angular areas of alternately higher and lower values of the scalar (fig. 47), and so on.

The said features of the field give the practical rules for drawing the curves. Examining the numbers we first look for the points where the scalar has its extreme values. Around these points we then draw closed curves, proceeding subsequently to the curves representing intermediate values of the scalar and having the more complicated course between the areas of greater and those of smaller values. Among these curves the ones intersecting themselves should not be avoided. They give more information regarding the field than any other single curve.

It is important to observe the remarkable completeness of the graphical representation of a scalar function. When we draw the equiscalar curves for unit intervals, these curves will represent not only the scalar itself, but also its ascendant or its gradient (section 17). Any one of these vectors gives complete information regarding the result of any differentiation of the first order performed upon the scalar. The drawing of the equiscalar curves involves therefore a differentiation of the scalar function. The representation gives not only the function itself, but also its differentials. We shall derive great advantage from this later, when we have to perform differential operations in graphical form.

127. The Drawing of Vector-Lines.—The drawing of vector-lines by the use of arrows representing observed directions of a vector and the drawing of equiscalar curves by the use of observed values of a scalar are analogous operations, inasmuch as interpolations have to be performed by eye-measure. But in one case the interpolations are of scalar nature, in the other of vector-nature, interpolations of direction.

This difference regarding the nature of the interpolations is intimately related to a difference of principle between the two operations: The drawing of equiscalar curves involves a differentiation in graphical form of a scalar function; the drawing of the vector-lines involves an integration in graphical form of a differential equation, namely, the differential equation for the vector-curves. We have performed the corresponding analytical integrations in special cases above (sections 120, 122, 123). This graphical integration would not contain any difficulty if arrows of absolutely correct direction completely covered the plane of the drawing. But the curves have to be drawn by the use of the minimum of data given by the observations, and with attention paid to the limited accuracy, or to the direct errors of the observations. Under these circumstances, in order to get the lines drawn as correctly as possible, it will be important to make as complete use as possible of the general properties of the field. We must derive from them qualitative rules which allow us to make the correct use of the data contained in the observations.

For this we shall have to pay special attention to the *singularities* of the field, *i. e.*, to the mutual intersections and touchings of the lines of flow; for as soon as the places are determined where intersections or touchings take place, and as soon as the manner is known in which the lines of flow pass through these places, the general feature of the field will to a great extent be given; for everywhere else in the field the lines will be limited in their course by the condition of not cutting or touching each other.

128. Simplest Singularities in the Field of the Lines of Flow.—We have chosen our examples in the preceding chapter so as to illustrate the simplest singularities which can arise in the three-dimensional solenoidal field; and forming the horizontal sections through these fields we have seen the character of the corresponding singularities in the two-dimensional vector-fields which we shall use to represent the three-dimensional one. In the simple cases treated analytically, the fields had simple properties of symmetry. Drawing correspondingly crooked and

asymmetric figures, we get pictures of the singularities and of the field surrounding them as they will appear in the case of concrete motions. In this manner we get the schemes of singularities presented by the different diagrams of fig. 48. The following remarks regarding each of them will easily be understood by a comparison with the results obtained analytically in sections 120, 122, 123 of the preceding chapter.

I. *Neutral Points.*—Points of this description appear when opposite currents meet each other and bend off against each other without producing any motion normal to the sheet (section 120). In the singular point two lines of flow will intersect each other. Points of higher order, in which a greater but still finite number of lines of flow intersect each other under finite angles, are also theoretically possible (fig. 38), though they will occur rarely.

II. *Points of divergence and of convergence.*—Let a field in space as that of fig. 42 (p. 37) be given. The corresponding two-dimensional field contained in a horizontal plane is represented by fig. 43 D. It contains a point in which an infinite number of lines of flow intersect each other. A tangential motion of this kind in a sheet always depends upon the existence of a motion normal to the sheet, leading masses into it or taking masses away from it. In the atmospheric sheet near the ground a point of divergence will appear where there is a descending current (centre of anticyclone) and a point of convergence where there is an ascending current (center of cyclone). The lines of flow are drawn in diagrams B–E of fig. 48, with the common spiral-formed curvature due to the earth's rotation, which is so well known from the air-motions near the centers of cyclones or anticyclones. In the sheet of water near the sea's surface a point of divergence will depend upon an ascending motion and a point of convergence upon a descending motion of the water masses below. When the sheet is situated at a greater distance from the bounding surfaces, divergence in the tangential motion shows that the normal motion brings greater masses into the sheet on one side than it brings out on the other, and vice versa for convergence in the tangential motion. But no definite conclusion can be drawn regarding the general direction of this normal motion, which may even have opposite directions on the two sides of the sheet.

III. *Lines of divergence and of convergence.*—Let a field in space, as that described in section 123, be given. Fig. 45 D shows that the two-dimensional field in a horizontal plane will contain a singular line of flow from which an infinite number of other lines of flow diverge out asymptotically (fig. 48 F). Reversing the direction of the motion, we get a similar line toward which an infinite number of lines of flow converge asymptotically (fig. 48 G). Evidently the lines of divergence and convergence are in precisely the same relation to the normal motion as the points of convergence and of divergence. In the case of rapid convergence, the designer can make no difference between common and asymptotical touching. When the singular line is represented by a stroke of finite breadth, it will completely absorb the lines converging toward it. The case of an infinitely rapid convergence arises when the lines go normally into the singular line, the case $A_y = 0$ or $b = 0$ in the example of section 123. In this case the asymptotic line ceases to be a line of flow and is reduced to be a line for zero numerical value of the vector.

The singularities presented by the lines of flow are in a definite relation to the field of intensity. As we have remarked already, wherever vector-lines intersect each other under finite angles, the vector must have the numerical value zero. In the same

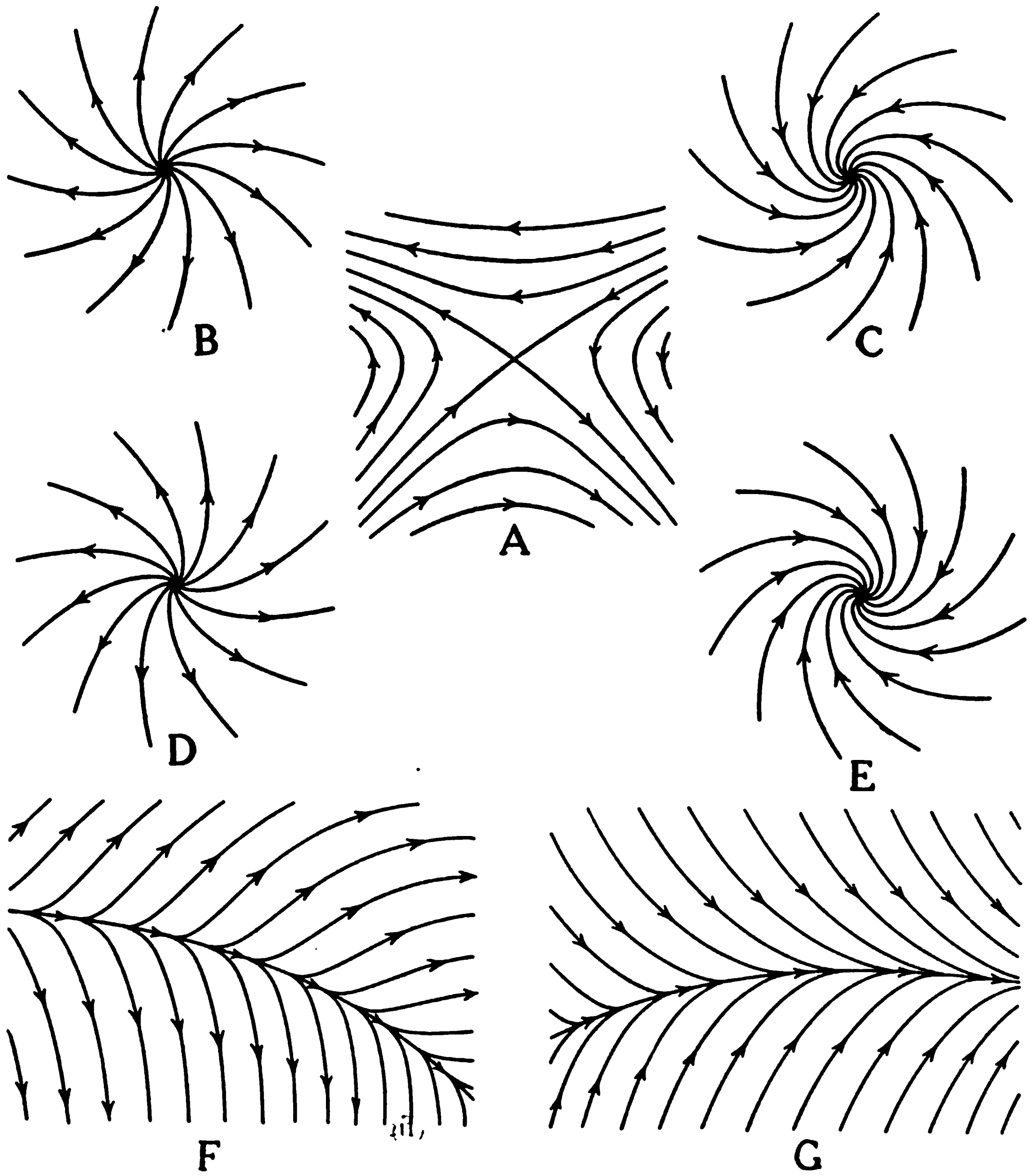


FIG. 48.—Simplest singularities in two-dimensional vector-field.

A. Neutral point.
B. Point of divergence, northern hemisphere.
C. Point of convergence, northern hemisphere.
D. Point of divergence, southern hemisphere.
E. Point of convergence, southern hemisphere.
F. Line of divergence.
G. Line of convergence.

manner the vector must have smaller numerical values in the asymptotic lines than on both sides of it, because the components normal to the line disappear in the line. Thus:

> *The numerical value of the vector is zero in the singular points, and has a relative minimum in the singular lines.*

The curves of equal intensity must therefore be closed around the neutral points and around the points of convergence and divergence, and make a bend as they pass lines of divergence or of convergence. This bend may be very slight and impossible to discover by the observations when the lines of flow have a slow convergence toward the singular line. But in the case of rapid convergence the bend should come out strongly.

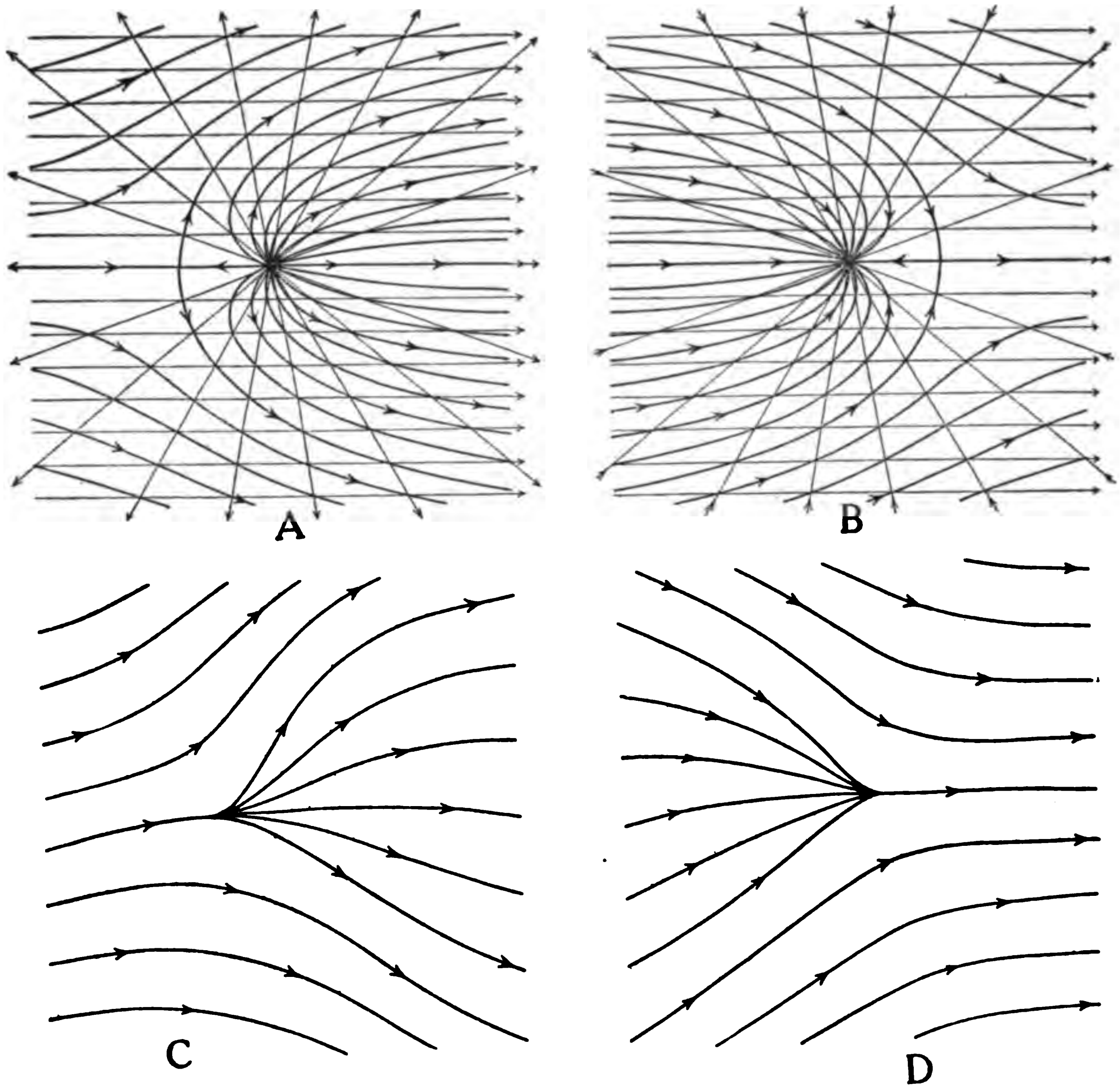


FIG. 49.—Complexes of singular points.

A. Neutral point and point of divergence.
B. Neutral point and point of convergence.
C. Line of flow branching out into several lines.
D. Lines of flow joining into one.

129. Complexes of Singular Points.—When good observations are at hand, it will generally cause no greater difficulty to discriminate the nature of the singular points as long as they are separated from each other by sufficiently large spaces; but it may be more difficult when singular points of different nature appear close

together. It will therefore be important to consider the conditions for the formation of some such complexes of singularities.

Let us for this purpose consider two coexistent fields, a simple field of translation represented by the parallel straight lines of flow, and a field containing a point of divergence, having the straight radial lines of flow of fig. 49 A. For the sake of simplicity we may consider also the last field as solenoidal except at the center itself, the normal supply being localized to this point instead of being spread over a finite area. Under these conditions we can add the solenoidal fields graphically (section 121). We then get the resultant field represented by the heavy lines of fig. 49 A, containing the constellation of two singular points, a point of divergence and a neutral point. Fig. 49 B shows the result of the same construction when the field of translation is retained, while the second field is changed into one containing a center of convergence. The field has the same character as the preceding one, only reversed.

This constellation of a point of convergence or divergence and a hyperbolic point will often occur on the charts of air-motion along the earth's surface. It appears as the result of a main horizontal wind and a vertical descending, respectively ascending, current. The discrimination of this constellation will cause no difficulty when the phenomenon is on a sufficiently large scale, and the two singular points are thus at sufficiently great distances from each other; but they may also get so near to each other that no observations of the air-motion is obtained between them. The direct drawing of the lines of flow from the observations will then give points or places where a line of flow branches out into several branches (fig. 49 C), or several lines of flow join into one (fig. 49 D). At the point of ramification the different branches may touch each other or cut each other under finite angles. The first case presumes a minimum and the second zero numerical value of the vector at this point.

130. Complex Phenomena in Connection with Lines of Convergence and of Divergence.—The theoretical possibility of certain complex singularities is seen at once. A line of convergence or of divergence can contain a neutral point in which the direction of the motion tangential to the line changes its sign (fig. 50 A, B). A line of divergence can come out from a point of divergence, and a line of convergence can end in a point of convergence (fig. 50 C, D). The latter seems to be no rare phenomenon in well-developed cyclones. Several lines of convergence are also often seen to join into one (fig. 50 E).

A specially interesting feature is the closed line of convergence containing within the inclosed area a point of divergence (fig. 50 F). This gives the kinematic aspect of the phenomenon called *eye of cyclone*, which seems to be common in strong cyclones. Corresponding eyes of anticyclone are also kinematically possible, though for dynamic reasons less probable.

A remarkable feature sometimes found on synoptical maps representing the air-motion along the ground is lines alternately of convergence and of divergence running more or less parallel to each other.

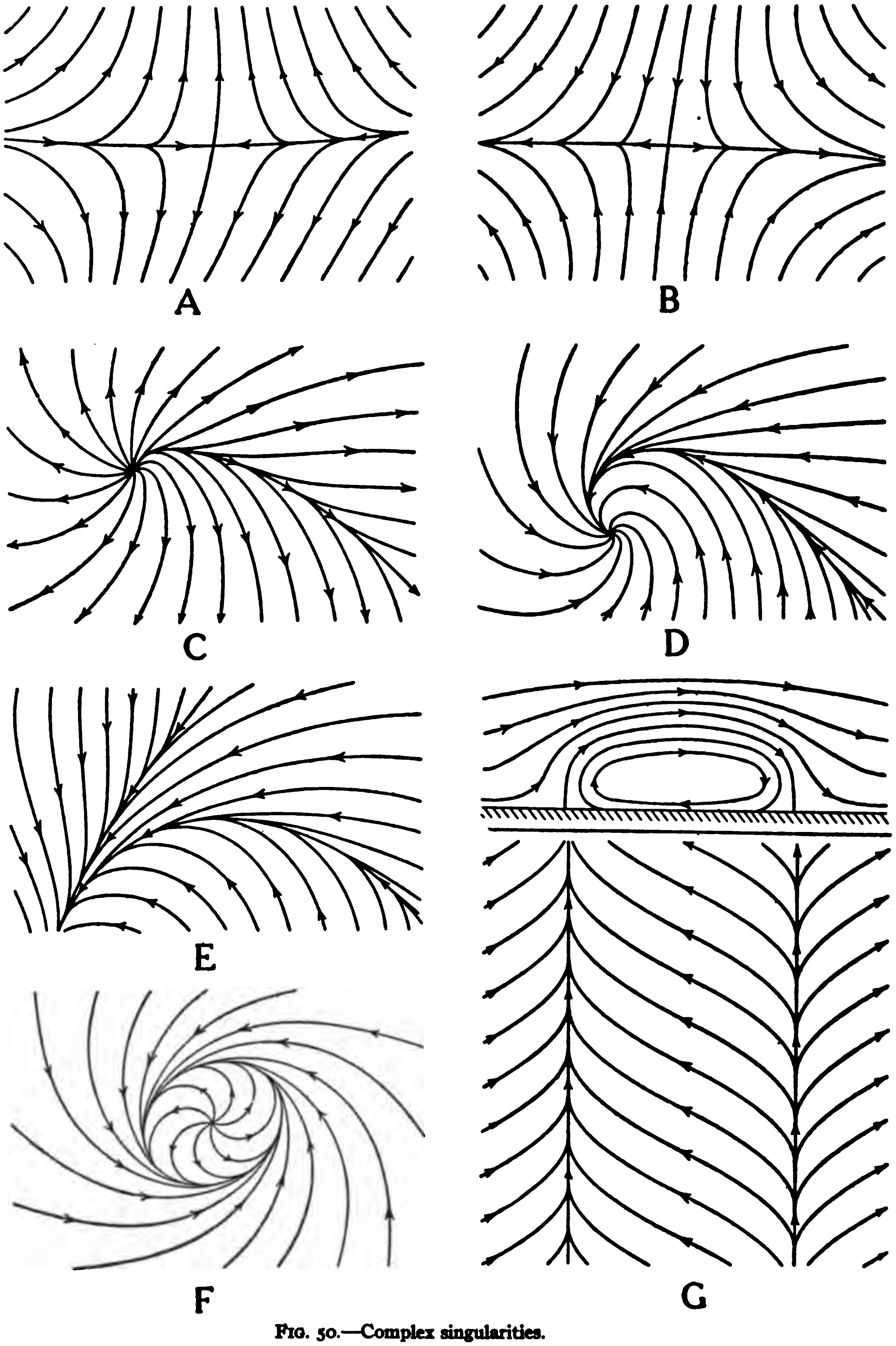


FIG. 50.—Complex singularities.

A. Line of divergence with neutral point.
B. Line of convergence with neutral point.
C. Line of divergence issuing from point of divergence.
D. Line of convergence ending in point of convergence.
E. Two lines of convergence joining into one.
F. Eye of cyclone.
G. Rolling mass of air bordered by a line of convergence and a line of divergence.

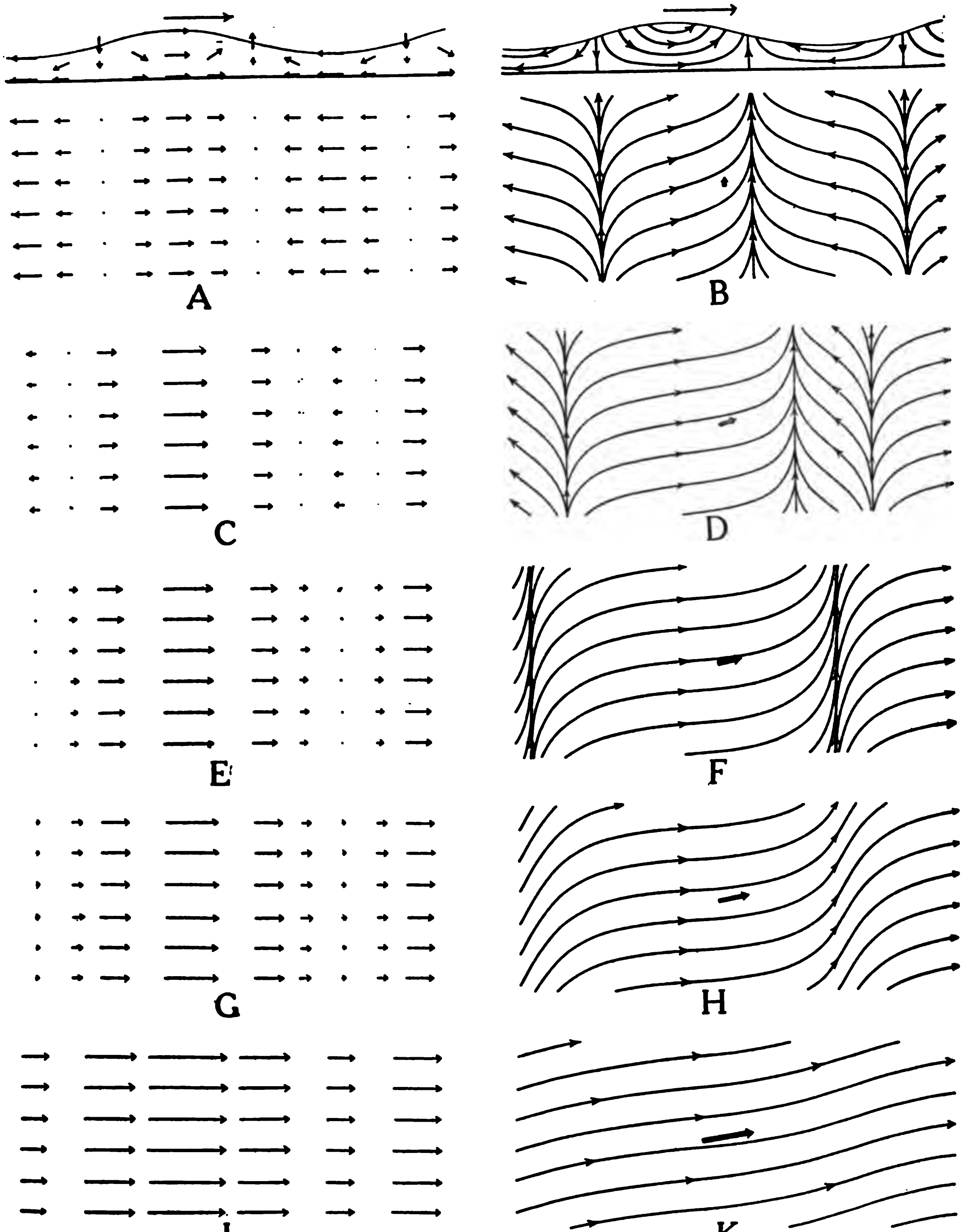


FIG. 51.—Effect of combined wave-motion and motion of translation.

A. Pure wave-motion.
B. Translation parallel to wave-ridges.
C. Translation normal to wave-ridges.
D. Translation oblique to wave-ridges.
E. Stronger translation normal to wave-ridges.
F. Stronger translation oblique to wave-ridges.
G. Still stronger translation normal to wave-ridges.
H. Still stronger translation oblique to wave-ridges.
I. Still stronger translation normal to wave-ridges.
K. Still stronger translation oblique to wave-ridges.

The corresponding motions in space may be of different kinds. Thus a rolling mass of air (fig. 50 G) will be bounded by a line of convergence and a line of divergence parallel to each other. But the most common origin of such lines may be wave-motions.* We shall therefore examine this case separately.

131. Influence of Wave-Motions on the Aspect of the Lines of Flow.—The large-scale waves which can arise in the atmosphere will be of the same nature as long waves in shallow water. During the propagation of the waves the different particles will describe elliptic orbits in vertical planes normal to the wave-ridges. Every ellipse has its long axis horizontal and its short axis vertical. The latter axis will decrease as we go downward, and be zero at the ground. Thus the motion near the ground will consist in rectilinear oscillations.

Remembering the difference of phase from particle to particle, we can draw arrows representing the *simultaneous* motion of a set of particles at a given epoch. This distribution of arrows in a vertical plane is shown at the top of fig. 51 A, and the corresponding lines of flow at the top of fig. 51 B. As will be seen, the propagation of the waves depends upon a conflux of masses below the front-slope and a corresponding afflux below the back-slope of the waves. In the horizontal projection we shall therefore always get a line of convergence below the front slope and a line of divergence below the back-slope of every wave. These lines will follow the waves in their motion of propagation.

Fig. 51 A will thus give the instantaneous distribution of motion at the ground in the case of a pure wave-motion. With the system of velocities thus given we shall compose the constant velocity due to a pure translation.

(1) First let us add a constant velocity which is parallel to the direction of the wave-ridges. Performing the parallelogram-constructions and afterwards drawing the lines of flow, we get the picture of fig. 51 B. The picture shows lines of flow running between a system of parallel and equidistant asymptotic lines, alternately lines of convergence and of divergence.

(2) To the velocities of fig. 51 A we shall now add a constant velocity which is normal to the direction of the wave-ridges and of smaller intensity than the greatest velocity due to the pure wave-motion. We shall then get the picture of fig. 51 C. When we afterwards add the same constant velocity parallel to the direction of the wave-ridges as above, perform the parallelogram-constructions, and draw the lines of flow, we get the picture of fig. 51 D. The picture shows parallel, but no more equidistant, lines of convergence and divergence.

(3) To the velocities of fig. 51 A we shall again add a constant velocity of direction normal to the wave-ridges, but now of intensity equal to the greatest occurring in the pure wave-motion. We shall then get the velocities presented by fig. 51 E. When we add in this case the same constant velocity parallel to the wave-ridges as above, perform the parallelogram-constructions, and draw the lines of flow, we shall get the picture of fig. 51 F. Here we have a set of wave-formed lines of flow, touching

*Cf. J. W. Sandström: Ueber die Beziehung zwischen Luftdruck und Wind. K. Svenska Vetenskapsakademiens Handlingar, T. 45, No. 10. 1910.

each other along a set of singular lines, each produced by the coincidence of a line of convergence and a line of divergence.

(4) To the velocities of fig. 51 A we shall finally add a constant velocity of direction normal to the wave-ridges and now of greater intensity than the greatest velocity due to pure wave-motion. We then get velocities which are periodically increasing and decreasing, but without any change of direction (fig. 51 G). If we add to these velocities the same constant velocity parallel to the direction of the wave-ridges as above, we shall get the system of asymmetric wave-formed lines of fig. 51 H, containing no singularity.

(5) If we increase still further the velocity normal to the wave-ridges, and then add the same velocity parallel to the wave-ridges as before, we shall get fig. 51 I and K respectively. The lines of flow of the latter figure are very nearly sinus-lines, but of very small amplitude.

In all of the figures B, D, F, H, K, the velocity parallel to the wave-ridges has the same value, and a very small value. If we increase this velocity, the lines of flow of the figures B, D, F will be stretched out in the direction of the singular lines, *i.e.*, in the direction of the wave-ridges, and the lines of flow of the figures H and K will get higher waves.

132. **Practical Rules for the Direct Drawing of the Lines of Flow and the Curves of Equal Intensity.**—When a chart is given containing arrows and numbers representing the observations of the motion, the first thing to do in order to pass on to the continuous representation of the motion will be this: by examination of the distribution of arrows and of the corresponding intensities to find out the nature and the approximate situation of the singularities.

This being done, it will generally be best first to draw certain of the lines of flow issuing from the singularities. Some lines of flow will generally be found whose course can be drawn with great certainty. A set of such lines being drawn, the general character of the whole field will practically be determined, for they will divide the chart into areas within which the other lines must have their course, as intersections are excluded except in the singularities.

The lines of flow and those of equal intensity should be drawn with continuous attention to each other. The closed intensity-curves surrounding the singular points are first drawn, then other closed curves surrounding other places of maximum or of minimum values of the vector, and then by and by the curves which have a more complicated course.

In this way, it will generally not be found too difficult to draw the lines of flow and curves of equal intensity, representing the air-motions along the ground over the areas where we have a satisfactory network of meteorological stations. Cases of doubt as to the character of the singularities as well as to the detailed course of the curves may arise. But making the experiment of letting different workers draw the curves of flow from the same observations independently, we have always found that the result has been very nearly the same as soon as the observations have the completeness of those from Europe or from the United States.

CHAPTER VI.

SUPPLEMENTARY RULES TO ASSIST IN THE DRAWING OF THE LINES OF FLOW AND OF THE CURVES OF EQUAL INTENSITY.

133. **Remarks on the Digression.**—We have emphasized the fact that the drawing of the lines of flow and of the curves of equal intensity would cause no difficulty, if we had at our disposal a sufficient number of really good observations; but as a matter of fact the observations are often so scarce and so heterogeneous that great doubts arise as to the course of the lines. In such cases we must look for other diagnostic methods than the pure kinematic ones.

This leads us to give here, in anticipation, diagnostic rules depending upon dynamic, partly also upon thermodynamic and other principles. The foundation of these rules will be considered more fully in later parts of this work. Deviating thus for practical reasons from the strictly theoretical plan, it will be important to make certain reservations in connection with this digression.

If the aim be simply this, to find the most probable motion of atmosphere or hydrosphere on a certain occasion, it is perfectly legitimate to bring into application all diagnostic methods which may serve the purpose. But if further conclusions should be drawn from the picture of motions thus obtained, we must take care to avoid the *circulus vitiosus*. If rules derived from dynamic or thermodynamic principles have been used to produce the picture of atmospheric motions, this picture can not be used legitimately afterwards to verify these same rules.

It can not therefore be too strongly recommended to develop the system of direct observations of atmospheric and hydrospheric motions, in order to make it possible to arrive at the synoptical representations of the motions by methods of a purely kinematic nature. Representations obtained in this way will be the only ones which can be legitimately used for subsequent investigations regarding the dynamic and thermodynamic phenomena which are the causes of the motions.

134. **Relation of the Kinematic Singularities to Dynamic and Thermodynamic Phenomena.**—Motion has a general tendency to go from higher toward lower pressures. From this we easily derive the following special rule:

Within a barometric depression there is a probability for existence of points or lines of convergence; within areas of high pressure there is a probability for the existence of points or lines of divergence. Long ridges of high pressure will as a rule contain a line of divergence; long ridges of low pressure a line of convergence. In the neighborhood of a maximum-minimum point of pressure situated between two high and two low areas there will be a probability for the existence of a neutral point with hyperbolic lines of flow.

Where the given observations of the wind do not give sufficient evidence for the nature and placement of the singularities, the required supplementary evidence may be obtained by examining the chart of pressure. But in doing so we should remember that there is no necessity for the motion to go always, and under all con-

ditions, toward lower pressure. There will seldom be an absolute coincidence between the points of convergence or of divergence with the points of minimum or of maximum pressure, or between the neutral point and the saddle point on the isobaric surfaces. The draftsman will often find that the observations of the wind give full evidence for the existence of kinematic singularities, especially of neutral points and of lines of convergence and of divergence at places where the chart of pressure does *not* show the expected peculiarities. Examples where the pressure for theoretical reasons shows other peculiarities will be considered below.

For thermodynamic reasons the kinematic singularities are in similar relation to the distribution of precipitation, cloudiness, and blue sky, as to that of pressure. Within an area of precipitation or of cloudiness there is, as a rule, ascending motion and therefore a probability of the existence of a point or of a line of convergence. In the same manner within areas of blue sky there will usually be descending motion and therefore a probability for the existence of a point or a line of divergence. The neutral point, which has no relation to vertical motion, will be indifferent in its relation to precipitation and to blue sky.

The charts of precipitation, of cloudiness, and of blue sky may therefore be used precisely as those of pressure, to get additional evidence in cases where the observations of the wind are not sufficient. But as in the case of pressure, the conclusion can not be reversed. Especially there will often be found lines of convergence causing no precipitation. Examining the relation of the kinematic singularities to pressure and precipitation, cloudiness and blue sky, it will probably be possible to decide whether the singularity is a local one, concerning only the lowest strata, or whether it has any connection with the motion also at greater heights.

135. Consequences of the Stability of Atmospheric or Hydrospheric Equilibrium.—The different layers of the air or the sea as a rule rest upon each other in stable equilibrium. A mass of air or of water will not leave its level except it be forced to do so. The currents will therefore always prefer to some extent to go round instead of going over obstacles. In other words, the lines of flow will have a certain tendency to follow the level curves representing the topography of the bounding surfaces. Many striking examples of this are seen on the accompanying maps of the air-motion. This dependency of the wind-direction upon topography is so strong that it can be recommended to draw the lines of flow on outline-maps containing a simplified representation of the topography of the land. In many cases the apparent irregularity in the distribution of arrows representing the observed wind-directions will be understood at once, by a comparison with the level curves of this map.

Sea-motions will depend upon the configuration of the bottom still more than air-motions on the configuration of the ground. The remarkable correspondence of lines of equal salinity, or equal temperature,* even at the surface of the sea, with

*In his paper "Some oceanographic results of the expedition with the 'Michael Sars', 1900" (Nyt Magasin for Naturvidenskab, T. 39, Christiania, 1901), Professor Nansen says, p. 153: "If we consider the chart (Plate I) of the surface-salinity and temperature it must strike one how almost exactly the most saline surface-water follows the deepest channel of the Norwegian sea, and how the isotherms especially of 10° C. and 9° C. seem to be deflected in a way similar to the isobaths." Further observations on this and allied subjects are found in the same author's "Oceanography of the North Polar Basin," pp. 260 et seq. (The Norwegian North Polar Expedition, 1893–96, Scientific Results, Vol. III, Christiania, 1902), and in Helland-Hansen and Nansen: The Norwegian Sea, Chapter X, p. 311 (Christiania, 1909).

the course of bathymetric curves several thousand meters below is a very striking sign of this dependency.

Sudden disturbances of the equilibrium will give rise to wave-motions. There seems to be good evidence for the existence of large-scale waves in the bounding surfaces between different layers in the sea.* Motions of the same kind are equally possible in the atmosphere, and lines of flow of the character described in section 131 seem to show that they actually occur. When the motion has this character, we have no right to expect a minimum of pressure along the lines of convergence and a maximum of pressure along lines of divergence. In case of pure wave-motions, the maxima of pressure should be under the summits and the minima under the troughs of waves, while the line of convergence is under the front-slope and the line of divergence under the back-slope of the advancing wave. If a progressive motion is added, displacements of the lines of convergence and divergence take place, and their relation to the pressure will not be easy to see on the chart.

136. Consequences of Kinetic Instability, Discontinuous Motions, and Eddies.— A kinetic phenomenon which is equally well known, though not so well understood

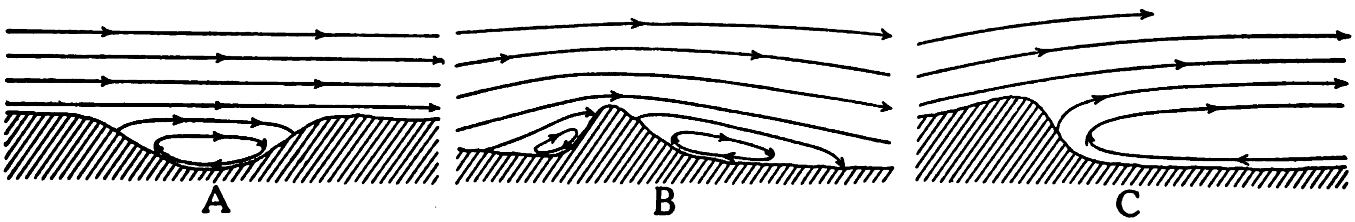


FIG. 52.—Motions due to kinetic instability.

A. Eddy in a valley. B. Eddies on windward and leeward side of a mountain. C. Eddy joining the great atmospheric motions.

as that of the formation of waves, is that of the formation of eddies. They are very often formed in the neighborhood of obstacles, both on the windward side of the obstacles, and still more frequently behind them. A motion going on without eddies is in such cases kinematically possible, but dynamically unstable, and has therefore no chance of persisting even if it be produced for a moment.

An eddy due to the instability of the motion may fill a valley across which there passes a main wind (fig. 52 A). It may be produced both on the windward and on the leeward side of a mountain (fig. 52 B), the latter case being the most frequent. The observations of the wind at the ground will then give pictures like that of fig. 50 G, with a parallel line of divergence and of convergence, the latter being as a rule the one which appears most distinctly. The line of divergence may also disappear completely when the eddy enters as a part of great atmospheric motions (fig. 52 C). In such cases only a line of convergence will be discovered following the ridge of a chain of mountains or the edge of a plateau-land. Eddies having a vertical axis may be formed in the same way. This latter kind of eddy will be very frequent in the atmosphere and perhaps still more so in the sea.† The eddies can exist on

*Regarding this question on submarine waves, cf. Helland-Hansen and Nansen's work just quoted, Chapter VI. See also V. W. Ekman's paper, "On Dead Water" (The Norwegian North Polar Expedition, 1893–96, Scientific Results, Vol. V, Christiania, 1906).

†Concerning eddies of large scale in the sea, cf. figs. 2, 37, 39, 105–107 of Helland-Hansen and Nansen's work just quoted; and especially pp. 311–312.

every scale, down to the smallest, which must be considered as local disturbances. These local eddies in connection with the sheltering effect of mountains and the deviating effect of valleys make the use of wind-observations from mountainous regions difficult. For such regions it would be good to have special information as to the peculiarities of each station, *i. e.*, to know the relation of the observed local wind to the general wind to be found higher up, where the influence of the obstacles is reduced or has disappeared. Signs representing these peculiarities could be introduced on the outline maps. The best method of investigating these peculiarities would be by sending up simultaneously from all stations pilot-balloons, giving the motions in the free air with which the local motions at the ground should be compared.

137. **Cold Wave, Warm Wave.**—Let us suppose a certain mass of air has been cooled down below the temperature of other masses in the same level. Equilibrium will then be disturbed, and in order to restore it the cool and heavy air will tend to spread out along the ground, driving away the warmer and lighter masses previously covering it. These will on the other hand go up, in order to fill the space from which the heavy masses of air sink down. In this case there will appear at the ground a line of convergence a little before the front of the advancing cold wave (fig. 53).*

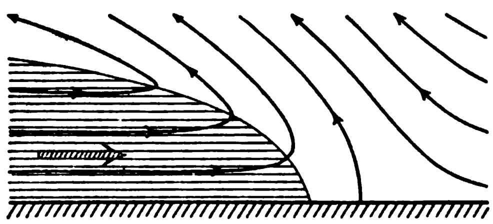


FIG. 53.—Cold wave.

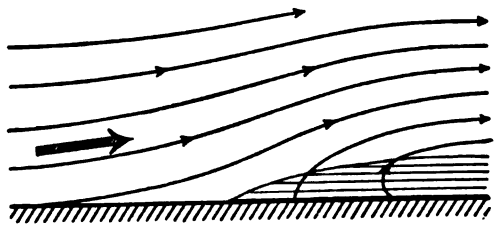


FIG. 54.—Warm wave.

Let us consider, on the other hand, a warm mass of air resting originally in hydrostatic equilibrium upon a thin sheet of cooler air. This arrangement will be stable as long as there is no motion or only a feeble motion. But if the upper layer has a sufficiently strong motion, the arrangement will be kinetically unstable. The warm air will then roll up and sweep away the thin layer of cool air. In this case there will arise at the ground a line of convergence a little before the front of an advancing warm wave (fig. 54).

In such cases there is no reason to expect a minimum of pressure along the line of convergence. There may come a sudden change of pressure as the line passes, but the most striking effect will be the sudden change of temperature along the line, and such a discontinuity of temperatures may give additional evidence for the existence of a line of convergence when the wind-observations themselves are insufficient.†

138. **Lines of Convergence at the Sea's Surface.**—While the observations of the motions themselves are difficult at sea, the situation of a line of convergence will under favorable circumstances be strikingly visible, for the reason that all sorts

*Cf. Sandström's paper, quoted p. 54.

†Cf. R. G. K. Lempfert and Richard Corless: Line squalls and associated phenomena. Quarterly Journal of the Royal Meteorological Society. London, April, 1910.

of floating objects, such as foam, seaweed, wood, etc., are collected in this line. Such lines are seen on a small scale near the shores when the wind is directed against the land. They then run parallel to the shore, often only like an oily band, marking the limit between the somewhat brackish water near the shore and the more salt water outside. Mr. Sandström has investigated directly the motion in the neighborhood of this line and found horizontal and vertical motion to be that represented by fig. 55.* Under the same condition of wind against the coast these lines exist on greater scale several kilometers from the coast, separating the coast-water from the salter sea-water. They are very well known by the fishermen, especially on account of the danger to the nets when they are set out across the line. These lines may also be seen under favorable circumstances on the open ocean, separating sea-currents of opposite directions.†

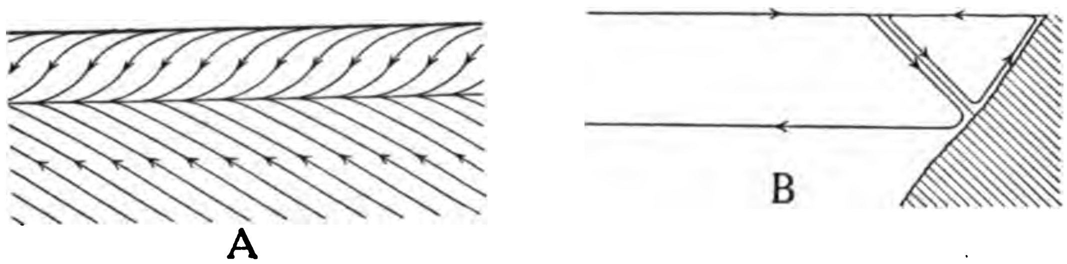


FIG. 55.—Line of convergence at sea.
A. Motion at sea's surface. B. Motion in a vertical section.

The investigation of these lines, their course, the degree of their constancy, etc., may be of great use for the kinematic investigation of the oceans.

139. Dynamic Diagnosis of Motion in the Free Space.—The observations of the air-motion in the higher strata are still too scarce to form the basis of a satisfactory construction of the motions, if only direct kinematic methods should be used. The arrows on the charts are far too few to determine the course of the lines of flow, and the numbers added to them are too few to determine the course of the lines of equal wind-intensity. We can not therefore avoid relying upon dynamic principles, if in such a case as this we should be able to give a fairly probable reconstruction of the air-motion on this occasion.

*J. W. Sandström: Windströme in Gullmarfjord. Svenska Hydrografisk-biologiska Kommissionens Skrifter II.

†Although the phenomenon must often have been observed, not only near the coasts, but also in the open sea, I have not been able to find any reference to it in literature in the latter case. I am indebted to Professor Fridtjof Nansen for the following communication concerning this case:

"Lines of convergence, as you mention, are frequently met with in the open sea, wherever a surface-current formed by light surface-water meets with another current formed by heavier water. Such conditions are quite common along the margin of the East-Greenland Polar Current. I remember especially to have observed such a remarkably distinct line of convergence in the Denmark Strait, northwest of Iceland, in about 66° 42′ N. Lat. and 26° 40′ W. Long. where we were with the 'Michael Sars' on August 3, 1900. The cold but light surface-water of the Polar Current met here with the warmer but more saline and consequently heavier water of the Irminger Current, coming from the south. One could distinctly see how the latter water flowed in under the surface-layer of polar water, and everything floating on its surface was, as it were, skimmed off by the polar water, especially of course all kinds of foam, and the line of convergence between the two currents was consequently marked with quantities of this foam which had been skimmed off, and we could thus easily trace the line across the sea surface, as far as the eye could reach toward the horizon, both northeastward and southwestward."

Setting aside on the one hand frictional resistance, and on the other hand the acceleration of the particles of air, we get a motion determined dynamically by the equilibrium between pressure-gradient and deviating force of the earth's rotation. Recent observations have shown that the true motion in the higher strata is usually not very different from that determined by this equilibrium condition.* The ideal motion existing when this condition is fulfilled is directed along the level curves on the isobaric surfaces and goes on with a velocity represented by the formula

$$v = \frac{1}{2\,\omega\, a \sin\varphi}$$

ω is the angular velocity of the earth, measured in radians per second ($\omega = 0.000073$); φ is the latitude, and a the distance in meters between level lines corresponding to unit difference of level (one dynamic decimeter). The difference of level between the successive curves being on some of our charts 10, on others 50 dynamic meters, we can use the formula

$$v = \frac{100}{1.46\, a \sin\varphi}, \text{ or, for the greater interval, } v = \frac{500}{1.46\, a \sin\varphi}$$

measuring the distance a between the curves in millimeters on our chart in the scale 1 : 10 000 000.

To use this principle to complete the observations on the charts, we have first constructed the level curves for the isobaric surfaces representing a pressure equal to the arithmetical mean of the pressures at the upper and the lower limits of the sheet. These curves are easily found by the principle of graphic addition, by drawing the diagonal curves through the parallelograms formed by the curves of absolute topography of the lower and the relative one of the upper bounding surface of the sheet, after having left out every second of the last curves.

The accordance of these curves with the direction of the arrows is never complete, and should be complete only in exceptional cases. Drawing the lines of flow (fig. B of the plates LVII–LX) we have made them cut the level lines under angles similar to those under which the arrows cut them (fig. A of the same plates). Further, the numbers representing the observed wind-intensities are never in full accordance with the formula. We have drawn the curves of equal wind-intensity (fig. B of the mentioned plates) so as to get departures from the theoretical value similar to those presented by the observations as seen by fig. A of the same plates.

Of course, many different drawings of the lines of flow and curves of intensity can be produced which are in accordance with these elastic rules. To what degree we have succeeded in reconstructing by plates LVII–LX, the true horizontal motion within each sheet will therefore remain an open question. We can not therefore too strongly recommend further work to produce satisfactory direct observations of atmospheric motions. Provisionally, the synoptic representations in higher strata which we have obtained will serve our nearest aim, viz, that of illustrating formally the further steps in the work of kinematic diagnosis.

*E. Gold: Barometric gradient and windforce. London, 1908.

CHAPTER VII.

ISOGONAL CURVES.

140. Isogonal Curves.—The drawing of vector-lines from the observed directions of a vector is an operation of the nature of an integration (section 127). On account of the incompleteness of the observations this integration is combined with interpolations. But it will be possible to separate from each other these two heterogeneous operations of interpolation and of integration. This is obtained by the method of isogonal curves devised by Mr. Sandström.*

We have agreed to represent observed directions by numbers (section 98). Instead of inscribing the arrows we can inscribe these numbers on a chart. Then we can draw curves joining the points where these numbers are equal. In all points of such a curve the vector will have the same direction, *i. e.*, form the same angle with the north-south line. These curves may therefore be called *isogonal* curves or *isogons*.

A chart containing these curves may be considered a completely interpolated representation of the differential equation determining the vector-curves. This representation being obtained, the integration will cause no difficulty. Across each isogonal curve we can draw short lines of the direction represented by the curve. These will be line-elements of the vector-lines. In this manner we can get the whole plane filled with such line-elements, and joining them to continuous curves we get the vector-lines.

141. Singular Points in the Field of a Multiple-Valued Scalar.—The isogonal curves represent the field of a multiple-valued scalar, the angle. The angle has no true greatest and no true smallest value. From the highest number, 64, used in our representation, we interpolate to the lowest, 1; for 1 represents the same angle as 65 would do.

In order to see the consequences which this peculiarity of the scalar has on the appearance of the field, let us suppose observations to have been taken at the points of a closed curve and to have given in succession the numbers from 1 to 64; in this case isogonal curves representing all angles must run in through the closed curve, in order to cut each other somewhere in the area contained within it. The point of intersection will be a *singular point*.

In the diagrams of figs. 56 and 57 the isogonal curves passing through the singular point are for the sake of simplicity drawn as straight radii. The numbers belonging to these radii may be arranged in two different ways: they can increase in the

*J. W. Sandström: Ueber die Bewegung der Flüssigkeiten. Annalen der Hydrographie und der maritimen Meteorologie. Berlin, 1909.

same direction as the numbers on the dial of fig. 32, the singular point will then be called *positive;* or in the opposite direction, the singular point will then be called *negative.* The eight diagrams of fig. 56 represent positive, the two of fig. 57 negative singular points, the successive diagrams are differing from each other by the situation of the initial isogon, that represented by 0 or 64. The change from

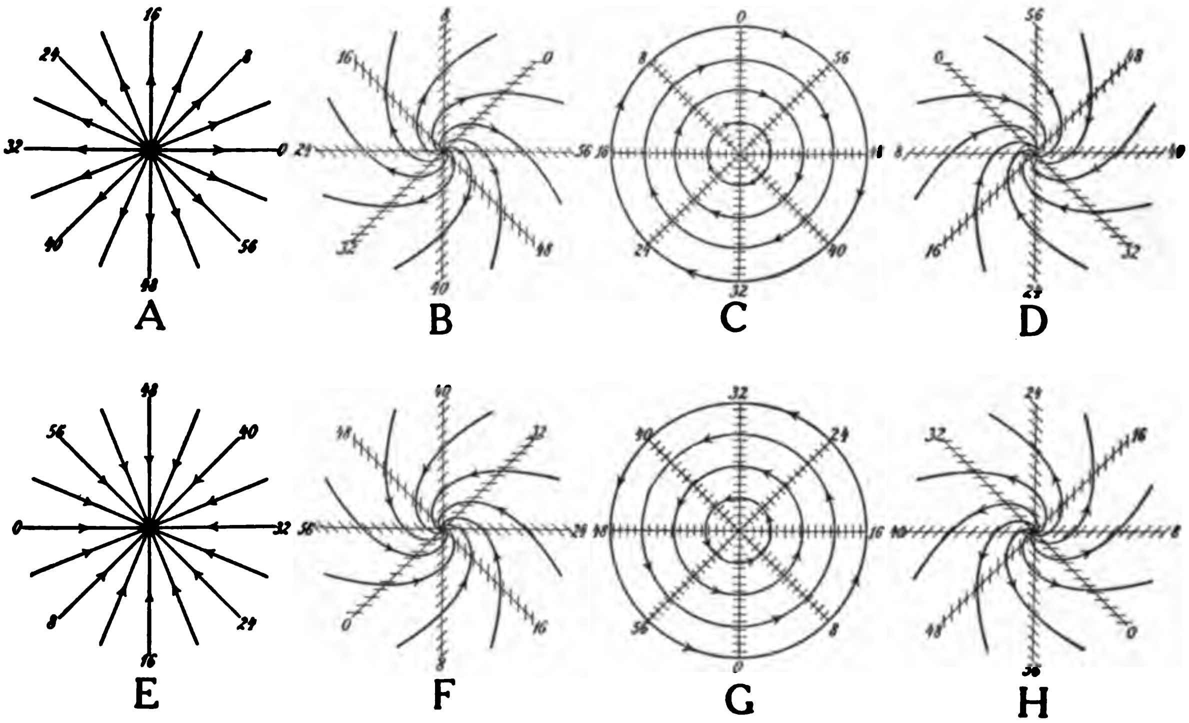


FIG. 56.—Positive singular points of isogonal curves.

A. Pure divergence.
B. Anticyclonic spirals of northern hemisphere.
C. Anticyclonic circles of northern hemisphere, cyclonic of southern hemisphere.
D. Cyclonic spirals of southern hemisphere.
E. Pure convergence.
F. Cyclonic spirals of northern hemisphere.
G. Cyclonic circles of northern hemisphere, anticyclonic of southern hemisphere.
H. Anticyclonic spirals of southern hemisphere.

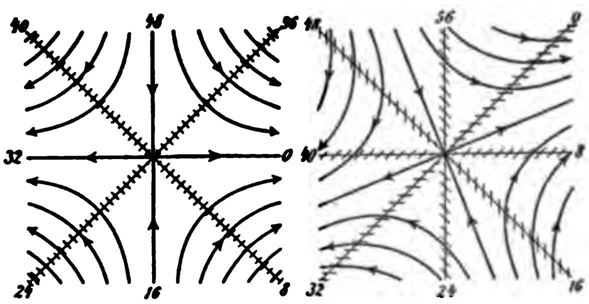


FIG. 57.—Negative singular points of isogonal curves.

diagram to diagram represents a *rotation of the system of isogons* of 45°. In all cases we can draw the short lines across the isogonal curves, and then the vector-curves. The diagrams will then show the features of the vector-field, which in the different cases corresponds to the singular point in the field of isogonal curves.

The examination of the figures leads to the following results:

(1) *The positive singular point of isogons corresponds to a point of divergence or convergence, the negative singular point to a neutral point of the vector-field.*

(2) The rotation of the system of isogons of a positive point has as a consequence that the vector-lines take the form of spiral curves of all types, including the limiting cases of straight radial lines and of circles.

(3) The rotation of the system of isogons of a negative point has as a consequence a rotation of the system of hyperbolic vector-lines without any change in their form; the angle of rotation of the vector-lines is half as great as that of the isogonal curves.

When the isogonal curves are no longer straight radii with constant angular intervals, but curves with more irregular intervals, the vector-lines of the corresponding vector-field will no longer be true logarithmic spirals or true hyperbolæ; but otherwise the character of the field will remain unchanged. If the numbers 1 to 64 are repeated twice or a greater number of times on a contour surrounding the singular point, always increasing in the same direction, the singular point will be of higher order. Only the negative singular points will be physically possible; but even they will occur rarely and be of small practical interest. (Cf. fig. 38.)

142. Further Remarks on the Field of Isogonal Curves and their Relation to the Vector-Field.—When the isogonal curves are to be drawn, the first thing will be to discover the situation of the singular points. For this we have to examine whether closed contours can be found on which the numbers representing the observations always increase in the same direction. If this be the case we are sure that there must be a singular point within the contour. As these singular points will always coincide with the singular points of the vector-field, we can also find these points by the use of rules which we have developed in the preceding chapters.

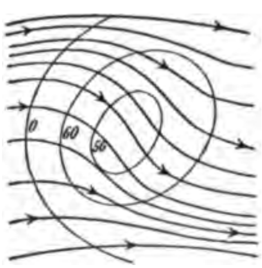


FIG. 58.—Closed isogonal curves. Inflexions of vector-lines.

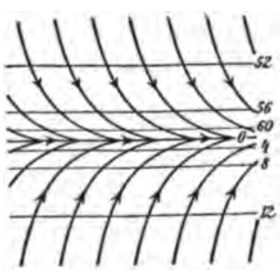


FIG. 59.—Parallel isogonal curves.

FIG. 60.—Concentric circles as isogonal curves.

The situation of the singular points being found, in which the curves intersect each other, the drawing of the curves will involve no other difficulties than those connected with the drawing of the equiscalar curves of the single-valued scalars; for isogonal curves representing different angles can never intersect each other in other points. Besides curves issuing from or entering into the singular points there will be found closed curves surrounding places of what may be called maxima or

minima. Within these regions the lines of flow will have points of inflexion (fig. 58). As in the fields of the single-valued scalar, there may appear complexes of such maxima and minima, containing between them a maximum-minimum point where a certain singular isogonal curve cuts itself (fig. 46).

It is remarkable that no special singularity of the isogonal curves corresponds to lines of convergence or of divergence in the field of motion. Fig. 59 shows a case where such lines appear in the case of rectilinear and parallel isogonal curves, fig. 60 a case where they appear in the case of circular concentric isogonal curves. The feature of the isogonal curves in the case of the wave-motions described in section 131 is remarkably simple. Let the numbers on the rectilinear and parallel isogonal curves oscillate between two extreme values for instance, 52 and 12. If the isogonal curves run parallel to the average wind-direction, we get the parallel and equidistant lines of convergence and divergence of fig. 61 A. As the angle between the average

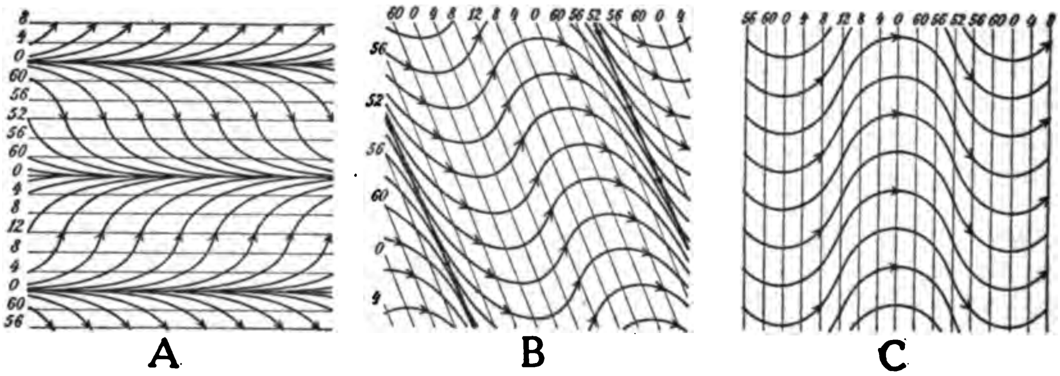


FIG. 61.—Isogonal curves for combined wave-motion and motion of translation.

A. Isogonal curves parallel to the main wind-direction.
B. Isogonal curves oblique to the main wind-direction.
C. Isogonal curves normal to the main wind-direction.

wind-direction and the isogonal curves increases, the singular lines are displaced relatively to each other, until finally two and two join into one, as in fig. 61 B. For still smaller angles we get sinusoidal lines of flow, the case of symmetry (fig. 61 C) arising when the isogonal curves are normal to the main wind-direction.

143. Sandström's Integration-Machines.—Mr. Sandström has based a method for graphical integration of differential-equations upon the representation of these equations by isogonal curves.* These curves being drawn, the tracing of the curves representing the integral, *i. e.*, the vector-curves, will cause no difficulty. Still, the draftsman will find it time-wasting to measure out the precise angles which these curves will have as they pass the different isogonal curves. But the work of drawing the vector-lines is very much facilitated by special machines constructed by Mr. Sandström, which trace automatically line-elements of the required direction across the isogonal curves. The construction of these machines will depend upon the system of coordinates to which the angles are referred. If the angles are referred to the meridians of a chart drawn in conical projection, very simple devices may be used. Fig. 62 shows a simple instrument serving the purpose in this case. A rule

*See note, p. 63.

R can slide through a guide which can turn around the pivot *P*. This pivot is fixed at the point of convergence of the meridians of the chart. At its other end the rule carries a toothed wheel *W*, which may be fixed in a position, where the edge of the teeth (*i. e.*, of the axis of the wheel) forms any given angle with the meridians. This angle is measured at the dial *D*. If the wheel is colored and carried along the isogonal curves, it will mark lines of the required direction across them.

During the motion the wheel partly slides and partly rolls. As the resistance against these two motions is not equal, it requires some care to follow precisely the

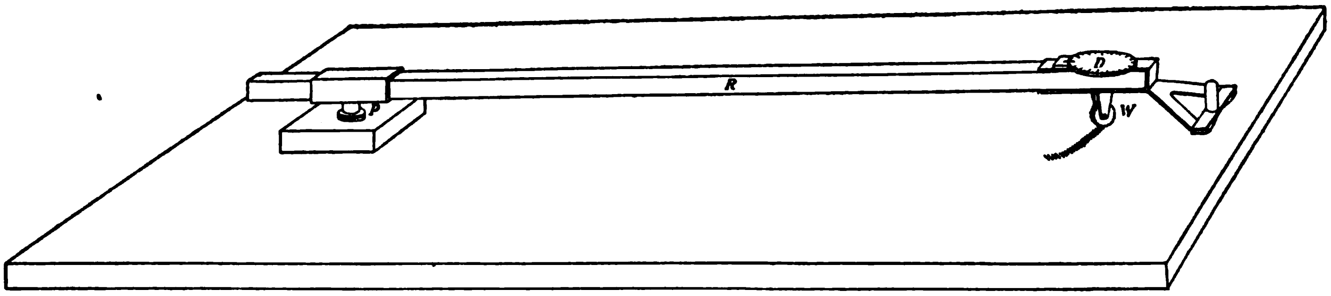


Fig. 62.—Machine for tracing line-elements across isogonal curves.

given curves. It will for this reason be advantageous to have an adjustable friction at the pivots of the toothed wheel. Fig. 63 shows another instrument by which this difficulty is avoided. Instead of a toothed wheel, the rule *R* carries a drum *D* with a caoutchouc membrane. This membrane carries a metal plate with a chisel *C*, which writes a line-element when it touches the paper. By an alternating air-current the membrane is set in motion, making the edge go up and down. When the chisel has this motion and is guided along the curve, it will mark the required

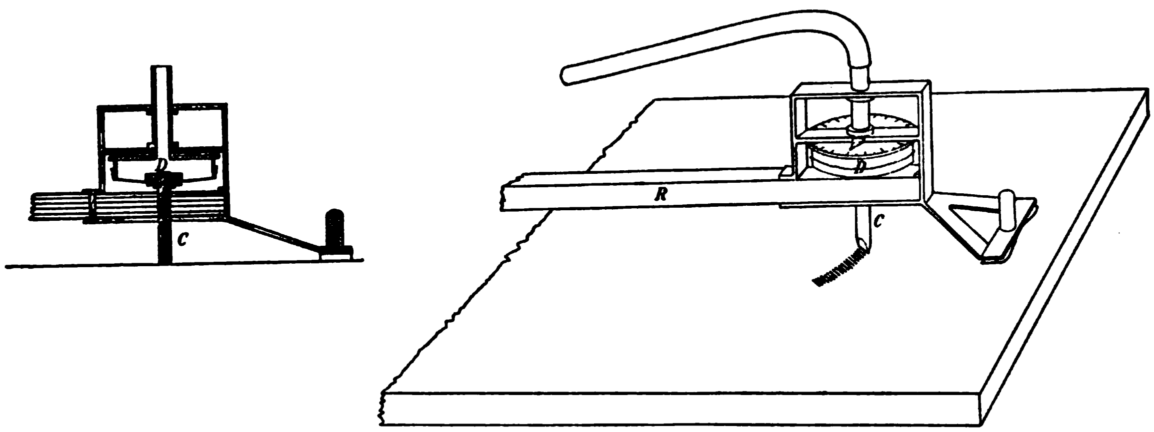


Fig. 63.—Other machine for tracing line-elements across isogonal curves.

line-elements across it. The desired angle with the meridian can be obtained by turning the drum, which on its upper face carries a dial with the required divisions. The alternating air-current for driving the membrane is obtained from another drum, joined with the crank of a rotating wheel, which is driven by a little electromotor.

When the charts are drawn on semi-transparent paper, no special device is required to color the tooth-wheel or the chisel. A coloring paper can be placed under the transparent sheet upon which the isogonal curves are drawn. The line-elements will then come on the under-side of the sheet, but will be seen through it.

When an instrument like one of these is at hand, it will be found very convenient to draw the lines of flow in the indirect way, using the isogons as auxiliary curves. Of course the indirect method will always require more time than the direct one. The latter will therefore be preferable for rapid work. But the indirect method gives a much higher degree of precision, and should therefore be preferred when the purpose is quantitative scientific investigations.

144. Equivalence of Isogons and Vector-Lines for the Representation of Vector-Fields.—We have introduced the isogons as auxiliary curves for tracing the vector-lines; but in reality they are perfectly equivalent to these lines for the representation of the field. We shall therefore have henceforth to reckon with two different representations of the vector-fields; by intensity-curves in connection with vector-lines, and by intensity-curves in connection with isogons. We have used the first consistently hitherto because it gives the most conspicuous picture. But our work will consist henceforth in the performance of mathematical operations upon the field, and these operations are in many cases performed more easily when the direction of the vector is given by the isogons. Therefore, in the following chapters, when we are going to study methods for performing elementary algebraic or infinitesimal operations upon the fields, we shall have to take into consideration the one method of representing the vector as well as the other, trying to utilize the special advantages of each of them.

CHAPTER VIII.

GRAPHICAL ALGEBRA.

145. **Graphical Mathematics.**—When the synoptical charts are found which can be derived directly from the observations, the further work for the diagnosis of present or for the prognosis of future states will consist in the performance of mathematical operations with the data given by these charts. The development of proper graphical methods for performing these operations directly upon the charts will be of the same importance for the progress of dynamic meteorology and hydrography as the methods of graphical statics and of graphical dynamics have been for the progress of technical sciences. The first serious problems of these graphical mathematics will present themselves as soon as we shall accomplish kinematic diagnosis by determining the vertical motions. Afterwards we shall meet with such problems continuously. This will therefore be the moment for taking a general view of the character of these problems and of methods to be used for solving them.

The problems will present themselves in this form: a chart or a set of charts is given, representing the fields of certain scalars or vectors. Another chart or set of charts is to be derived from them, representing the field of other scalars or vectors, which are defined as functions of the first by relations in finite or in infinitesimal form.

One way for the solution of such problems will always be open. We perform discontinuously, for a certain number of points, the operations defined by the relations. This gives the values of the required scalars or vectors in a certain number of points. By use of these values we draw the charts representing the new scalars or vectors, just as we draw such charts by use of the observations taken at a finite number of points. By following this method we give up the idea of continuous fields during the performance of the mathematical operations, in order to return to the fields as soon as the operations have been performed. We shall call this the *discontinuous* method.

But on the other hand it will be possible to find methods by which the idea of the field is never given up. The method will then consist in the continuous tracing of curves guided by the data contained on the given charts, and by the relations containing the implicit definition of the new charts. Every operation leads to a chart representing a field, and it will, as a rule, be necessary to pass through several auxiliary fields in order to arrive at the required fields. We shall call these methods *continuous*, and the development of them will be our main object.

146. **Drawing-Board.**—Certain practical arrangements should be mentioned at once. It will be impossible to draw all the different curves on one sheet of paper. They must be distributed on several sheets. But at the same time we must be able

to make different systems of curves simultaneously visible in their true mutual position, as if they had been drawn upon the same sheet of paper. Certain measures must be taken to attain this.

We have found it most convenient to draw the different charts upon sheets of semi-transparent paper, and to have at hand a special drawing-board. This board consists of a sheet of glass with a wooden frame and has a contrivance for producing illumination from below. This illumination is obtained most easily by an incandescent electric lamp. The sheets of paper should cover the glass completely. They can be fixed to the wooden frame by drawing-pins. The paper should be sufficiently transparent, or the illumination sufficiently strong, to allow us to have at least three sheets simultaneously upon the board, two containing given systems of curves and a third upon which the derived curves are drawn. The plates accompanying this book have been printed upon paper which we have found convenient for this kind of work.

147. Graphical Algebra with One Variable.—Let α be a scalar function represented by a chart of equiscalar curves. These curves are to be drawn for what we shall call "integer values" of the scalar, using the expression in a widened sense as a shortened expression for "integer values multiplied by a positive or negative power of 10." By a suitable change of units they will get integer values in the common sense of the word. It is required to find the equiscalar curves which represent in the same way the field of another scalar

$$\varphi = f(\alpha) \tag{a}$$

In this case a curve $\alpha =$ const. will also be a curve $\varphi =$ const. But the curves which represent integer values of α will as a rule not coincide with those which represent integer values of φ.

The discontinuous method of finding the curves for integer values of φ will be this: by direct calculation to find the values of φ in a certain number of points, and then to interpolate between them the points where φ has integer values. These points will give the placing of the curves for integer values of φ between those for integer values of α.

But we can give a continuous method of solving the same problem: We then solve equation (a) with respect to the *known* variable α,

$$\alpha = F(\varphi) \tag{b}$$

and construct an auxiliary table in which the values of α are tabulated for integer values of the argument φ. Thus

TABLE E.—*Table-scheme for graphical algebra with one variable.*

φ	0	1	2	3	4	5	6	7	8	9
0	α_0	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9
10	α_{10}	α_{11}	α_{12}	α_{13}	α_{14}	α_{15}	α_{16}	α_{17}	α_{18}	α_{19}
20	α_{20}	α_{21}	α_{22}	α_{23}	α_{24}	α_{25}	α_{26}	α_{27}	α_{28}	α_{29}

Table E shows at once for which values of a we shall get integer values of φ. We can then at once draw the equiscalar curves for integer values of φ in their proper places between the given equiscalar curves for integer values of a.

As an example we can consider the square of a given field, $f(a) = a^2$. Thus

$$(c) \qquad \varphi = a^2$$

Solving with respect to a we get

$$(d) \qquad a = \sqrt{\varphi}$$

a is tabulated for integer values of φ in table F.

TABLE F.—*Square-root table for passing from the field of a scalar to the field of its square.*

φ	0	10	20	30	40	50	60	70	80	90
0	0	3.2	4.5	5.5	6.3	7.1	7.7	8.4	8.9	9.5
100	10.0	10.5	11.0	11.4	11.8	12.2	12.6	13.0	13.4	13.8
200	14.1	14.5	14.8	15.2	15.5	15.8	16.1	16.4	16.7	17.0

This table shows that the curve $\varphi = 50$ coincides with the curve $a = 7.1$, curve $\varphi = 60$ with curve $a = 7.7$, and so on. Fig. 64 shows how by use of this information

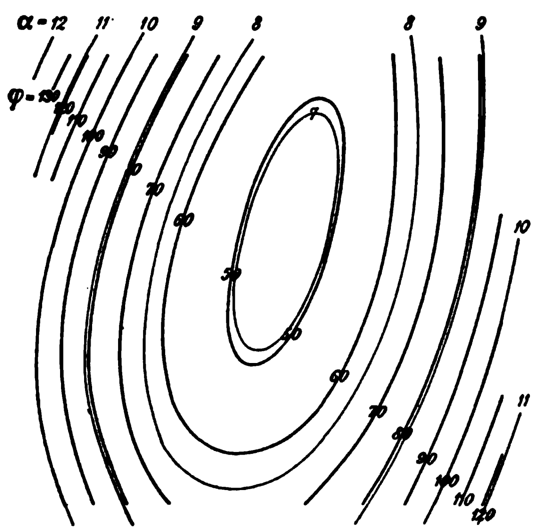


FIG. 64.—Field of a given scalar (fine lines $a = 7, 8, 9, \ldots$) and field of its square (thick lines $\varphi = 50, 60, 70, \ldots$)

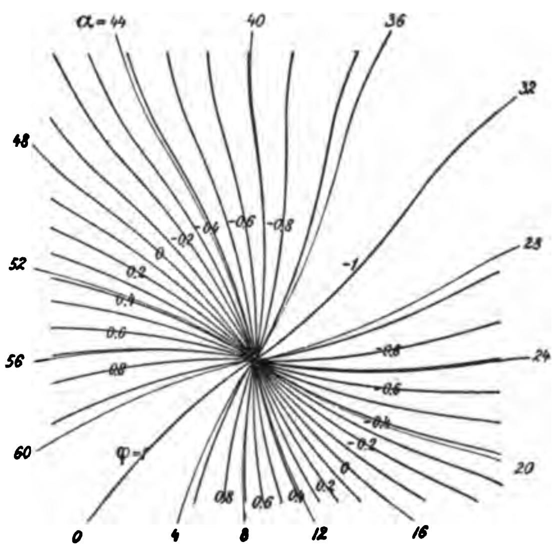


FIG. 65.—Field of an angle (fine lines $a = 0, 4, 8, 12, \ldots$) and field of its cosine (thick lines $\varphi = 1.0, 0.9, 0.8 \ldots$)

the curves for integer values of φ are drawn in their proper places between the curves for integer values of a. For evident reasons we have drawn the curves $\varphi =$ const. for ten times greater intervals than the curves $a =$ const.

To use another example, let the field of a multiple-valued scalar, the angle a, be given, expressed by the numbers 0–63. It is required to find the field of the scalar

$$(e) \qquad \varphi = \cos a$$

We then construct a table (table F′) according to the equation

(f) $$\alpha = \text{arc}\cos\varphi$$

By use of this table we can easily draw the curves for "integer" values of cos α between those for integer values of α (fig. 65).

TABLE F′.—*Arcus-cosine table for passing from the field of an angle to the field of its cosine.*

φ	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
	16	15.0	13.9	12.9	11.8	10.7	9.4	8.1	6.6	4.6	0
	48	49.0	50.1	51.1	52.2	53.3	54.6	55.9	57.4	59.4	0
φ	−0.0	−0.1	−0.2	−0.3	−0.4	−0.5	−0.6	−0.7	−0.8	−0.9	−1.0
	48	47.0	45.9	44.9	43.8	42.7	41.4	40.1	38.6	36.6	32
	16	17.0	18.1	19.1	20.2	21.3	22.6	23.9	25.4	27.4	32

148. Graphical Algebra with Two Variables.—Let α and β be two scalar functions, each represented by a chart of equiscalar curves. The problem is to draw the equiscalar curves representing the field of a third scalar φ, which is determined by the relation

(a) $$\varphi = f(\alpha, \beta)$$

The discontinuous method of solving this problem will be this: we choose a point, take out from the charts the values of α and β and calculate by equation (a), the corresponding values of $f(\alpha, \beta)$. This is repeated for a sufficient number of points. The values thus found for φ are inscribed upon a sheet of paper, and then the equiscalar curves $\varphi = 1, 2, 3, \ldots$ are drawn by leading of the values thus found. Evidently the work can be facilitated by the construction of an auxiliary table containing the values of φ tabulated with α and β as arguments.

But a corresponding continuous method can also be given. To see it we solve equation (a) with respect to *one of the known* quantities α or β,

(b) $$\beta = F(\alpha, \varphi) \quad \text{or} \quad \alpha = F'(\beta, \varphi)$$

According to these equations we construct the auxiliary tables G.

Let us first follow one of the vertical columns in the table and the corresponding curve α = const. on the chart. We shall then see that the curve $\alpha = 0$ will be cut by the curve $\varphi = 0$ at the point where β has the value β_{00}, by the curve $\varphi = 1$ at the point where β has the value β_{10}, by the curve $\varphi = 2$ at the point where β has the value β_{20}, and so on. The situation of the points $\beta_{00}, \beta_{10}, \beta_{20}, \ldots$ is seen at once, as the intersection of the curve $\alpha = 0$ with the curves $\beta = 0, \beta = 1, \beta = 2 \ldots$ shows where on the curve $\alpha = 0$ we have the integer values of β. Interpolating by eye-measure we can mark the points where the curves α = const. are cut by the curves for integer values of φ. These points being marked, we can draw at once the curves φ = const.

Instead of following the vertical columns we can also follow the horizontal lines of the table, and then draw directly one by one the curves φ = const., performing successively the interpolations by eye-measure which give the points of intersection with the different curves α = const. This method will usually be the most convenient.

TABLES G.—*Table-schemes for graphical algebra with two variables.*

φ	α 0	1	2	3	4	5
0	β_{00}	β_{01}	β_{02}	β_{03}	β_{04}	β_{05}
1	β_{10}	β_{11}	β_{12}	β_{13}	β_{14}	β_{15}
2	β_{20}	β_{21}	β_{22}	β_{23}	β_{24}	β_{25}
3	β_{30}	β_{31}	β_{32}	β_{33}	β_{34}	β_{35}

φ	β 0	1	2	3	4	5
0	α_{00}	α_{01}	α_{02}	α_{03}	α_{04}	α_{05}
1	α_{10}	α_{11}	α_{12}	α_{13}	α_{14}	α_{15}
2	α_{20}	α_{21}	α_{22}	α_{23}	α_{24}	α_{25}
3	α_{30}	α_{31}	α_{32}	α_{33}	α_{34}	α_{35}

The second table G can be used in precisely the same way to find the points of intersection of the required curves φ = const. with the given curves β = const. When φ is a symmetric function of α and β, the two tables will be identical with each other. Then one table will be sufficient, which may be provided with two sets of arguments, one set above and on the left side, the other below and on the right side. (Cf. tables H and I below).

We shall now make a few special applications of this general principle, taking the simplest algebraical operations, and giving the schemes for the construction of the most important auxiliary tables. More extensive tables will be given later in our collections of tables for practical use.

149. Addition of Scalar Fields.—Let the function be $f(\alpha, \beta) = \alpha+\beta$. That is, we shall determine the field of the scalar φ which is the sum of the scalars α and β

(*a*) $$\varphi = \alpha+\beta$$

The discontinuous method will consist in forming directly the sum $\alpha+\beta$ in a certain number of points, and to draw the equiscalar curves of φ by leading of these values.

In order to use the continuous method we write equation (*a*) in the form

(*b*) $$\beta = \varphi-\alpha \quad \text{or} \quad \alpha = \varphi-\beta$$

Both equations lead to the same table, table H, where on account of the symmetry we have an equal right to interpret α as argument and β as the tabulated quantity or β as argument and α as tabulated quantity.

The table shows that the curves representing the sum of the scalars α and β pass through the points for simultaneously integer values both of α and β as a set of *diagonal curves* (figs. 66 and 67), *i. e.*, we return to the simple process of graphical addition, of which we have made so frequent use. In this simple case the auxiliary table is superfluous. We have introduced it only to show the connection with the more complicated corresponding problems.

It will be seen at once that while the sum $\alpha+\beta$ is represented by the one set of diagonal curves, the difference $\beta-\alpha$ or $\alpha-\beta$ will be represented by the other set of diagonal curves.

TABLE H.—*Graphical addition.*

One addend tabulated as function of the sum and the other addend.

Sum. φ	First addend α.												
	1	2	3	4	5	6	7	8	9	10	11	12	
9	8	7	6	5	4	3	2	1	0	−1	−2	−3	9
10	9	8	7	6	5	4	3	2	1	0	−1	−2	10
11	10	9	8	7	6	5	4	3	2	1	0	−1	11
12	11	10	9	8	7	6	5	4	3	2	1	0	12
13	12	11	10	9	8	7	6	5	4	3	2	1	13
14	13	12	11	10	9	8	7	6	5	4	3	2	14
15	14	13	12	11	10	9	8	7	6	5	4	3	15
16	15	14	13	12	11	10	9	8	7	6	5	4	16
	1	2	3	4	5	6	7	8	9	10	11	12	φ Sum.
	Second addend β.												

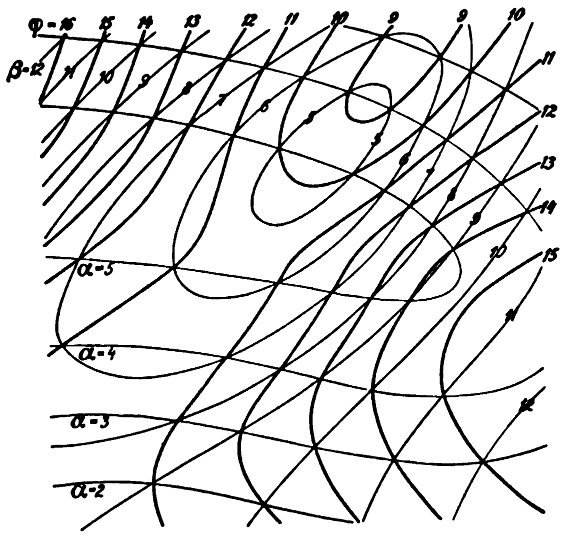


FIG. 66.—Graphical addition of single-valued scalar fields. (The fine lines represent the given fields, the thick lines their sum.)

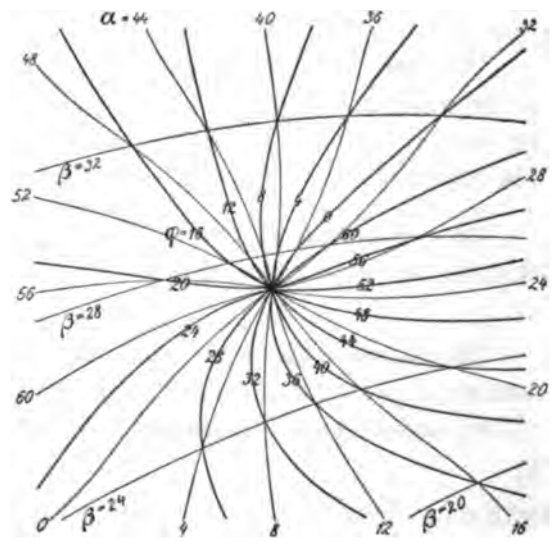


FIG. 67.—Graphical addition of multiple-valued scalar fields (fields of angles). The fine lines represent the given fields, the thick lines their sum. Observe the consequences of the multiple-values: $32+32=64=0$, $48+32=80=16$, etc.

150. Multiplication of Scalar Fields.—Let the function $\varphi=f\ (\alpha, \beta)$ be

(a) $$\varphi = \alpha\,\beta$$

In order to use the continuous method we solve with respect to β or α

(b) $$\beta = \frac{\varphi}{\alpha} \text{ or } \alpha = \frac{\varphi}{\beta}$$

TABLE I.—*Graphical multiplication.*

One factor tabulated as function of the product and the other factor.

Product. φ	First factor α.												
	1	2	3	4	5	6	7	8	9	10	11	12	
15	15.0	7.5	5.0	3.8	3.0	2.5	2.1	1.9	1.7	1.5	1.4	1.2	15
20	20.0	10.0	6.7	5.0	4.0	3.3	2.9	2.5	2.2	2.0	1.8	1.7	20
25	25.0	12.5	8.3	6.3	5.0	4.2	3.6	3.1	2.8	2.5	2.3	2.1	25
30	30.0	15.0	10.0	7.5	6.0	5.0	4.3	3.7	3.3	3.0	2.7	2.5	30
35	35.0	17.5	11.7	8.8	7.0	5.8	5.0	4.4	3.9	3.5	3.2	2.9	35
40	40.0	20.0	13.3	10.0	8.0	6.7	5.7	5.0	4.4	4.0	3.6	3.3	40
45	45.0	22.5	15.0	11.3	9.0	7.5	6.4	5.6	5.0	4.5	4.1	3.8	45
50	50.0	25.0	16.7	12.5	10.0	8.3	7.1	6.2	5.6	5.0	4.5	4.2	50
	1	2	3	4	5	6	7	8	9	10	11	12	φ Product.
	Second factor β.												

These two equations lead to the same table, table I, in which we are equally right in interpreting α as argument and β as tabulated quantity or β as argument and α as tabulated quantity.

Fig. 68 exemplifies the use of the table. In drawing, for instance, the curve for the constant value $\varphi = 30$ of the product, we use the line in the table which has the argument $\varphi = 30$. When we consider α as argument and β as the tabulated quantity, this line of the table tells us that the curve $\varphi = 30$ is to be drawn through that point of the curve $\alpha = 3$ where $\beta = 10$, through that point of the curve $\alpha = 4$ where $\beta = 7.5$, through that point of the curve $\alpha = 5$ where $\beta = 6$, and so on. If we consider β as the argument and α as the tabulated quantity, we see that the curve $\varphi = 30$ is to be drawn through that point of the curve $\beta = 12$ where $\alpha = 2.5$, through that point of the curve $\beta = 11$ where $\alpha = 2.7$, and so on.

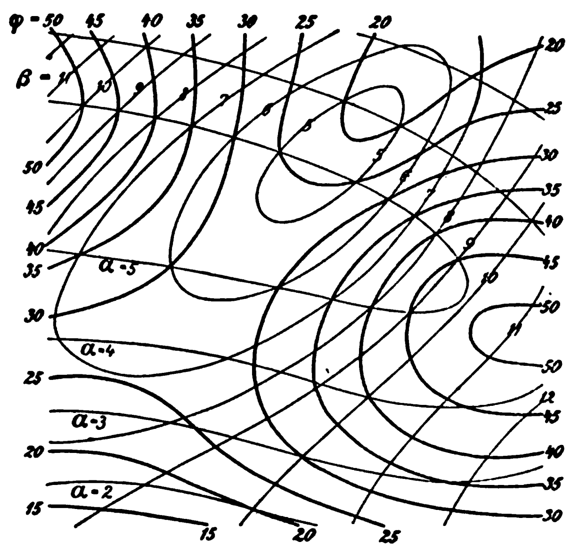


FIG. 68.—Graphical multiplication. The fine lines $\alpha = 2, 3, 4, \ldots$ and $\beta = 11, 10, 9, \ldots$ represent the factors, the thick lines $\varphi = 50, 45, 40, \ldots$ their product.

TABLES J.—*Graphical division.*

I. *Divisor tabulated as function of quotient and dividend.*

Quotient. φ	Dividend α.			
	2	3	4	5
0.2	10.0	15.0	20.0	25.0
0.3	6.7	10.0	13.3	16.7
0.4	5.0	7.5	10.0	12.5
0.5	4.0	6.0	8.0	10.0
0.6	3.3	5.0	6.7	8.3
0.7	2.9	4.3	5.7	7.1
0.8	2.5	3.8	5.0	6.3
0.9	2.2	3.3	4.4	5.6
1.0	2.0	3.0	4.0	5.0
1.1	1.8	2.7	3.6	4.5

II. *Dividend tabulated as function of quotient and divisor.*

Quotient. φ	Divisor β.							
	5	6	7	8	9	10	11	12
0.2	1.0	1.2	1.4	1.6	1.8	2.0	2.2	2.4
0.3	1.5	1.8	2.1	2.4	2.7	3.0	3.3	3.6
0.4	2.0	2.4	2.8	3.2	3.6	4.0	4.4	4.8
0.5	2.5	3.0	3.5	4.0	4.5	5.0	5.5	6.0
0.6	3.0	3.6	4.2	4.8	5.4	6.0	6.6	7.2
0.7	3.5	4.2	4.9	5.6	6.3	7.0	7.7	8.4
0.8	4.0	4.8	5.6	6.4	7.2	8.0	8.8	9.6
0.9	4.5	5.4	6.3	7.2	8.1	9.0	9.9	10.8
1.0	5.0	6.0	7.0	8.0	9.0	10.0	11.0	12.0
1.1	5.5	6.6	7.7	8.8	9.9	11.0	12.1	13.2

151. Division of Scalar Fields.—Now let $f(\alpha, \beta) = \frac{\alpha}{\beta}$. We shall then have to construct the field of the scalar φ, which is the ratio of the two scalars α and β,

$$(a) \qquad \varphi = \frac{\alpha}{\beta}$$

We here meet with the case that the function φ is asymmetric with respect to the two variables. Solving with respect to each of them we get

$$(b) \qquad \beta = \frac{\alpha}{\varphi} \qquad \text{or} \qquad \alpha = \beta\varphi$$

These equations lead to the two tables J. The first of them is the same as that serving graphical multiplication (table I), though other values of the arguments appear to suit the example of fig. 69. The second is an ordinary multiplication-table.

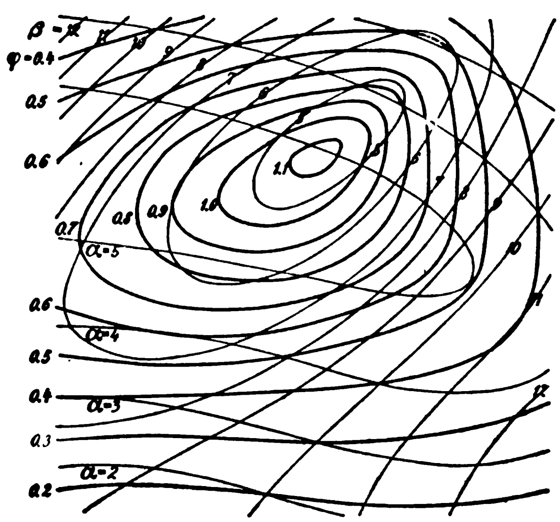


FIG. 69.—Graphical division. The fine lines $\alpha = 2, 3, 4, \ldots$ represent the dividend; the fine line $\beta = 12, 11, 10, 9, \ldots$ the divisor; the thick lines $\varphi = 1.1, 1.0, 0.9, \ldots$ the quotient.

The first of tables J shows for instance that the curve $\varphi = 0.6$ is to be drawn through that point of the curve $\alpha = 5$ where $\beta = 8.3$, through that point of the curve $\alpha = 4$ where $\beta = 6.7$, through that point of the curve $\alpha = 3$, where $\beta = 5.0$, and so on. The second table J shows in the same manner that the curve $\varphi = 0.6$ is to be drawn through that point of the curve $\beta = 9$ where $\alpha = 5.4$, through that point of the curve $\beta = 8$ where $\alpha = 4.8$, and so on. Observing thus the tabulated numbers, we can draw continuously one by one the curves $\varphi = $ const., which represent the required field.

152. Case of Three or More Variables.—Now let the scalar φ be a function of any number of variables

$$\varphi = f(\alpha, \beta, \gamma \quad . \quad . \quad . \quad)$$

In this case the discontinuous method, which consists of calculating the values of φ in any sufficient number of points and subsequent tracing of the equiscalar curves $\varphi =$ const., may be used precisely as in the case of two variables. But if we solve with respect to one of the given scalars, for instance α, in order to bring the continuous method into application, we meet with the practical difficulty connected with the tabulation of functions of more than two variables; for numerical tables can not easily be provided with more than two arguments.

In special cases it may be possible to decompose the complex operation into a series of partial operations each depending upon two variables only. Then all difficulties connected with the greater number of variables will drop out, and we can bring into application the methods which we have developed already, depending upon the construction of numerical tables with two arguments.

In the general case this decomposition of the problem will not, however, be possible. We must then look for other auxiliaries than numerical tables, and it will always turn out to be possible to produce special graphical or mechanical auxiliaries which will serve the same purpose as tables with more than two arguments would have done. These auxiliaries will, however, as a rule be more laborious to use than the tables with two variables. If, therefore, a reduction to problems with two variables is possible, it should generally be performed even if the number of single operations be thereby considerably increased.

We shall give the general method for constructing graphical tables which serve the purpose in the case when the number of variables is limited to three. Then let α, β, γ be three given scalar quantities. The field of each of them is represented by equiscalar curves. The problem is to find the equiscalar curves $\varphi =$ const., which represent the field

$$(a) \qquad \varphi = f\,(\alpha, \beta, \gamma)$$

In order to find the points of the curve $\alpha = \alpha_1$ in which φ has integer values, we have to examine the values of

$$(b) \qquad \varphi = f\,(\alpha_1, \beta, \gamma)$$

Here only β and γ are variables, and when we follow the curve $\alpha = \alpha_1$ (fig. 70B), we see that to any value of β will correspond a definite value of γ, and vice versa.

In order to find those values of one of them for which φ has integer values, we construct a graphical table. We set off β and γ as abscissa and as ordinate of a rectangular system of coordinates (fig. 70 A) and draw in this system of coordinates the curves $\varphi = 1, 2, 3, \quad . \quad . \quad .$ according to equation (*b*). We observe on the given chart (fig. 70 B) the values which β and γ have along the curve $\alpha = \alpha_1$. These values will define a certain curve in the system of coordinates β, γ. We draw this curve on a transparent sheet of paper, laid upon the graphical table fig. 70 A. This curve

will cut the curves $\varphi = 1, 2, 3, \ldots$ of the graphical table, and we can read off those values of β or of γ for which φ has integer values. Then we can set off these points along the curve $\alpha = \alpha_1$ on the given chart (fig. 70 B).

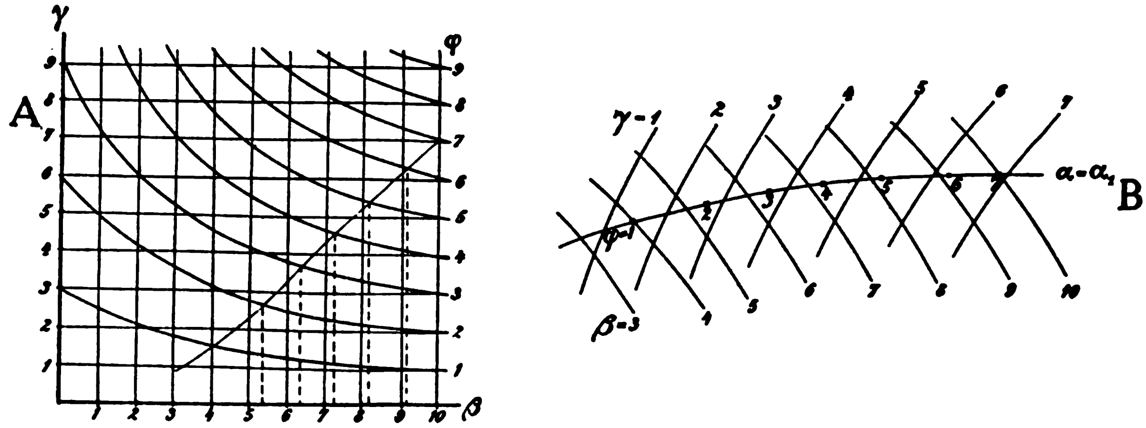


FIG. 70.—Example of graphical operations with three variables.

A. Scheme of graphical table.
B. α, β, γ, given fields. Construction of $\varphi = f(\alpha_1, \beta, \gamma)$.

If we construct a graphical table as that of fig. 70 A for each of the curves α = const., we can thus find a complete system of points determining the course of the curves φ = const.

153. Vector-Algebra.—It will be of special importance for us to bring graphical methods into application for mathematical operations concerning vector-fields. It will be useful and save circumlocution when at the same time we introduce a few simple notations of modern vector-analysis.*

A vector considered as a quantity which has both magnitude and direction will be denoted by a letter in heavy print. The corresponding letter in common print will denote its scalar value or tensor (intensity). The same letter in common print and with the suffix s will denote the projection of the vector on the direction s. In the same manner we shall by the suffixes x, y, z denote the projections on the three rectangular axes x, y, and z. Thus

Vector.	Tensor.	Projection on direction s.	Projections on rectangular axes.
A	A	A_s	A_x, A_y, A_z
B	B	B_s	B_x, B_y, B_z
. . .	. . .	. . . .	. . . .
F	F	F_s	F_x, F_y, F_z

*Compare: Gibbs-Wilson, Vector-Analysis, New York, 1901.

The fact that the vector **F** is the *vector-sum* according to the parallelogram-law of the two vectors **A** and **B** will be denoted by the vector-equation.

$$(a) \qquad \mathbf{F} = \mathbf{A}+\mathbf{B}$$

This equation can be considered as equivalent to the three scalar equations

$$(a') \qquad F_x = A_x+B_x \qquad F_y = A_y+B_y \qquad F_z = A_z+B_z$$

which express the projections of **F** as the scalar sum of the projections of **A** and of **B** (fig. 71). The scalar-sum of the tensors $A+B$ must be carefully distinguished from the scalar value or tensor $|\mathbf{A}+\mathbf{B}|$ of the vector-sum. There will be identity between the scalar sum of the tensors and the tensor of vector-sum when the two given vectors have the same direction, and between the scalar differences of the tensors and the tensor of the vector-sum when the two given vectors have opposite directions.

A scalar quantity which is equal to the product of the tensors of two given vectors and the cosine of the included angle will be called the *scalar product* of the two given vectors. When the given vectors are **A** and **B**, their scalar product shall be denoted by **A.B**, thus

$$(b) \qquad \mathbf{A}.\mathbf{B} = AB\cos\theta$$

By the fundamental formulæ of analytical geometry it is easily verified that the scalar product is equal to the sum of the products of the rectangular components of the given vectors,

$$(b') \qquad \mathbf{A}.\mathbf{B} = A_xB_x+A_yB_y+A_zB_z$$

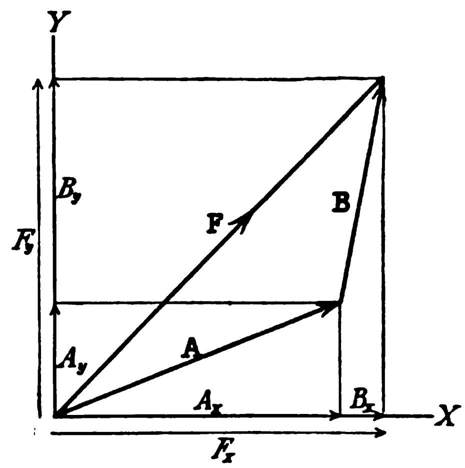


FIG. 71.—Vector-addition.

The vector-operations defined by the preceding formulæ are symmetrical with respect to the two given vectors **A** and **B**. In the vector-formulæ the symbols for the vectors can therefore be commutated

$$(c) \qquad \mathbf{A}+\mathbf{B} = \mathbf{B}+\mathbf{A} \qquad \mathbf{A}.\mathbf{B} = \mathbf{B}.\mathbf{A}$$

We shall define finally an important *unsymmetric* vector-operation, in which this commutation of the symbols will no more be allowed. The succession of the symbols will be used to serve an important purpose, namely, to distinguish between opposite directions in space. In order to give the definition of this operation, we must first make an important remark concerning the geometry of translations and rotations.

Let an axis in space be given. Two opposite translations will be possible along it, and two opposite rotations will be possible around it. We must agree upon a definite connection by which we can define the positive direction of rotation as soon as the positive direction of translation is chosen, and vice versa. We shall attain this by the rule of the *positive or right-handed screw*. When this screw moves in its

nut, it can not advance along its axis in a definite direction unless it performs a rotation around this axis in a corresponding definite direction; and vice versa it can not turn around its axis in a definite direction unless it advances along this axis in a corresponding definite direction (see fig. 72). Thus this screw connects a definite direction of translation with a corresponding definite direction of rotation, and vice versa. We shall agree to give the same sign to directions of translation and of rotation which are connected to each other in this way.

Two vectors in space, **A** and **B**, define two rotations which are smaller than two right angles, that from **A** to **B** and that from **B** to **A**. Both rotations take place around an axis which is normal both to **A** and to **B**, and can be represented symbolically by arrows pointing along the axis of rotation, in that direction which by the screw-rule is positive in reference to the direction of rotation.

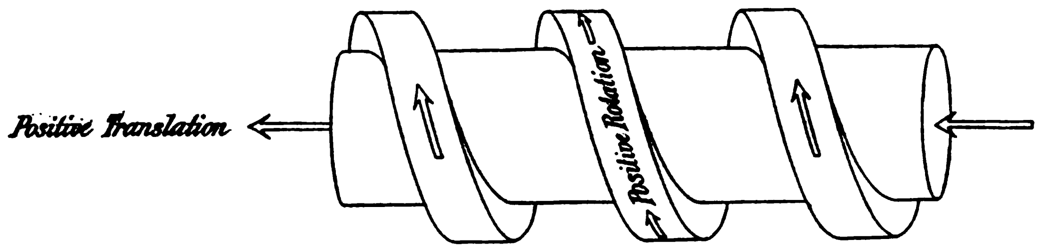


FIG. 72.—Positive-screw rule.

Now let us consider a vector **F** which is normal to the two given vectors **A** and **B**, which by its direction represents the rotation from **A** to **B**, and which has a tensor equal to the product of the tensors of the given vectors and the sine of the included

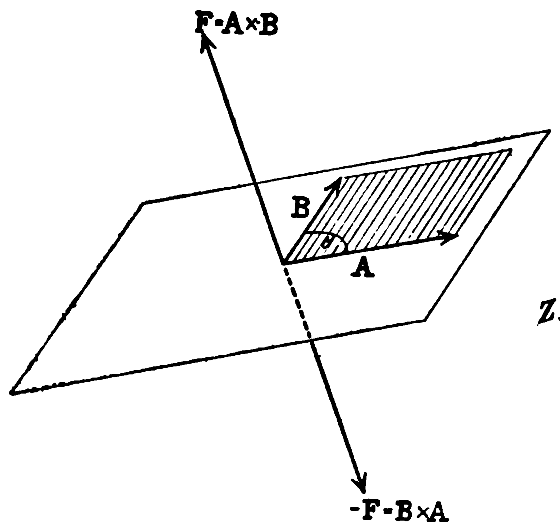


FIG. 73.—Vector-product.

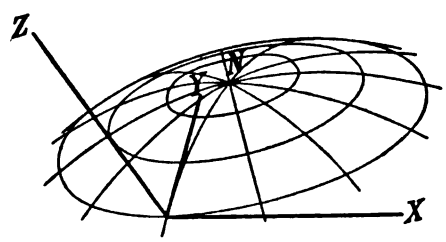


FIG. 74.—Positive system of rectangular coordinates.

angle. The fact that the vector **F** has this relation to the two vectors **A** and **B** will be expressed by the formula

(*d*) $$\mathbf{F} = \mathbf{A} \times \mathbf{B}$$

and **F** will be called the *vector-product* of the vectors **A** and **B**.

The relation of the vector-product **F** to the vector-factors **A** and **B** is illustrated by fig. 73: **F** is directed along the normal to the plane which contains **A** and **B**; the positive rotation around **F** transfers the first vector-factor **A** into the second **B**; and **F** has the scalar value F, which is given by the formula

$$F = AB \sin \theta \tag{d'}$$

or which is represented geometrically by the area of the parallelogram which has sides representing the vector-factors.

It follows immediately from the definitions that when we commutate the vectors **A** and **B**, we get the vector $-\mathbf{F}$, which is directed oppositely to **F**, thus

$$\mathbf{B}\times\mathbf{A} = -\mathbf{A}\times\mathbf{B} \tag{e}$$

When we bring coordinates into application we shall agree to use consistently what we shall call a *positive system of coordinates*. Let the positive direction along each of the rectangular axes be chosen. The corresponding positive rotation around an axis will then either be a rotation *in* or *against* that direction which is defined by the succession of letters X, Y, Z, X, . . . In the first case the system will be called positive, in the second case negative. Thus when the system is positive, the positive rotation around Z will go from X to Y, the positive rotation around X will go from Y to Z, the positive rotation around Y will go from Z to X. A positive system of coordinates, of which we shall make a frequent use, is one which has its axis of X directed toward the east, its axis of Y directed toward the north, and its axis of Z directed upward. (See fig. 74.)

When we use a positive system of coordinates, it is easily verified by the fundamental formulæ of analytical geometry that the rectangular components F_x, F_y, F_z of the vector-product **F** are

$$F_x = A_yB_z - A_zB_y \qquad F_y = A_zB_x - A_xB_z \qquad F_z = A_xB_y - A_yB_x \tag{f}$$

The vector-equation (d) may be considered as a shortened symbolic expression for the three equations (f). Equations (f) also at once lead to the result expressed by equation (e), that the vector-product changes its sign when the succession of the vector-factors is interchanged; for we get $-F_x$, $-F_y$, and $-F_z$ when in equations (f) we change A_x with B_x, A_y with B_y, and A_z with B_z.

154. Consistent Use of Rectangular Components in Graphical Vector-Algebra.— As drawings are two-dimensional, our methods can deal directly only with two-dimensional fields. Vector-fields in space must be treated indirectly. We have introduced for this the method of solving the three-dimensional field into fields tangential to and normal to a set of surfaces (section 118). The normal field may be treated as a two-dimensional scalar, while the tangential field represents a true two-dimensional vector. Our subject will therefore be that of developing graphical methods for performing mathematical operations upon these two-dimensional vectors.

One general method presents itself at once. We can introduce two sets of curves cutting each other under right angles, and use them as coordinate-curves. In the simplest case the two sets of curves will be two sets of parallel lines, which are mutually

perpendicular to each other. A vector is represented in every point of the field by its components along each of the two coordinate-curves passing through the point. The coordinate-curves are the vector-lines of the two vector-components. But as these vector-lines are given invariable curves which are common to the components of all vectors, no operations will have to be performed upon them. Although these components $\mathbf{A}_x$, $\mathbf{A}_y$, $\mathbf{B}_x$, $\mathbf{B}_y$, . . . are primarily vectors, we never need take into account their vector-nature. They will be represented completely by the fields of their scalar values A_x, A_y, B_x, B_y, ... The sign of the scalar value will give the direction of the component along the coordinate-curves. The graphical methods for scalar fields which we have developed will then come directly into application to all problems of vector-algebra.

When we follow this method, the problems of graphical vector-algebra are solved already.

Thus the *vector-sum* **F** of two vectors **A** and **B** will be represented by the two scalar components F_x and F_y, and each of them is found by graphical addition of the fields of the scalar components A_x and B_x, respectively A_y and B_y, in accordance with the equations

$$(a) \qquad F_x = A_x + B_x \qquad F_y = A_y + B_y$$

The *scalar product* of the two vectors **A** and **B** will be found by two graphical multiplications and one graphical addition in accordance with the formula

$$(b) \qquad A_x B_x + A_y B_y$$

In the case of the two-dimensional fields, the *vector-product* of two vectors will be normal to the surface which contains the field. From the point of view of two-dimensional geometry it therefore loses its character of a vector. We have to deal simply with a scalar

$$(c) \qquad A_x B_y - A_y B_x$$

and the field of this scalar is derived from those of four given scalars A_x, B_y, A_y, B_x by two graphical multiplications and one graphical subtraction.

The advantages gained by the consistent use of vector-components are great enough to make it a serious question whether it should not be favorable from the beginning to work exclusively with components, and not with the vectors themselves. From the point of view of the observations there will be no objection against this. It would be a good plan to observe separately the N.–S. and the E.–W. component of the wind or of the sea-motion. If the observations were taken with self-recording instruments, the vector-averages required (section 97) would be obtained by taking the ordinary average of each component separately. Neither would there be any objection from the point of view of the meteorological telegraphic service. Whichever system be used, two numbers will have to be telegraphed. In the one case the two numbers will have to represent the two rectangular components. In the other case one number must be used to represent the wind-intensity, and another to represent the wind-direction.

But as long as the observations are not very good and complete it may be a question if it be advisable to *draw* the charts for each component separately, without compounding them to a vector. The formal process of drawing equiscalar curves would be simple enough. But the difficulty would consist in smoothing out the irregularities and filling up gaps in the observations. This must be done with full understanding of the kinematical situation of which the true vector-chart gives a conspicuous picture, but the two separate component-charts present only a very imperfect picture. This full understanding of the situation will also be of use for the control when mathematical operations are to be performed on the charts. We shall therefore as a rule avoid the artificial representation of the vectors by two component-fields, and use as much as possible the direct representations.

155. Use of Angles to Represent the Directions of Vectors.—We have introduced two direct representations of the two-dimensional vector, by intensity-curves and vector-lines, and by intensity-curves and isogons. We shall as a rule prefer the latter when mathematical operations are to be performed. The angles which

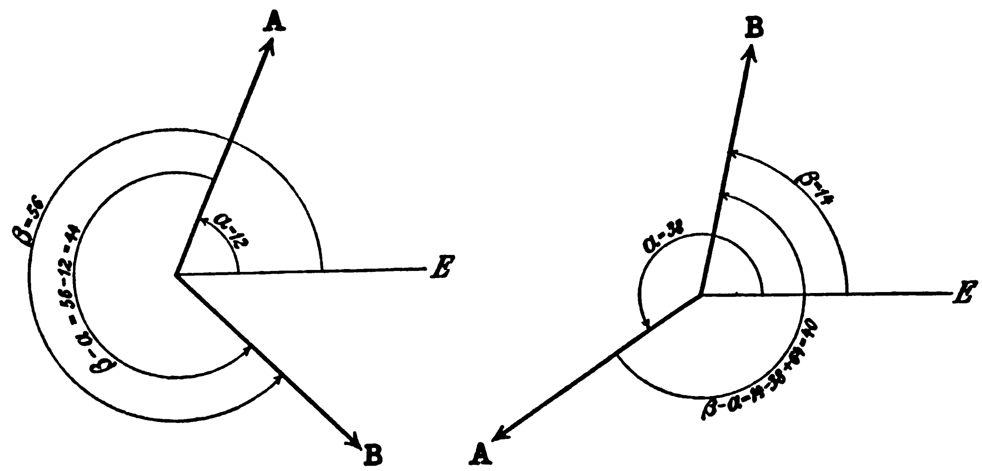


Fig. 75.—Angles and differences of angle.

represent the directions of the vectors **A**, **B**, . . . **F** will be represented by Greek letters $\alpha, \beta, \ldots \varphi$. We shall find it convenient occasionally in two-dimensional vector-algebra to use the symbols (A, α), (B, β), . . . (F, φ) as symbols for the vectors instead of **A**, **B**, . . . **F**. Thus we shall have identically

$$(a) \qquad \mathbf{A} = (A, \alpha), \ \mathbf{B} = (B, \beta), \ \ldots \ \mathbf{F} = (F, \varphi)$$

In order to define completely the angles $\alpha, \beta, \ldots \varphi$, we must agree upon the choice of *an initial direction* from which they should be counted, and *the direction of that rotation by which they should be produced.* The initial direction must be agreed upon by an arbitrary choice. On our charts in horizontal projection we will choose the direction toward E as this initial direction. The *positive rotation around a point*, or, what comes to the same, the *positive circulation around a closed curve,* is always to be defined in accordance with the positive-screw rule. Most of our charts will represent fields which are contained in horizontal or quasi-horizontal surfaces.

As we count the normal to these surfaces positive upward, the positive rotation around the normal will be a rotation against the motion of the hands of a watch and the positive circulation will be the cyclonic circulation E.–N.–W.–S. of the northern hemisphere. (Compare the dial of fig. 32.)

We shall agree to consider all given angles α, β, . . . *which are used to represent the direction of given vectors as produced by positive rotation from the chosen initial direction.* Thus all initially given angles will be represented by positive numbers which are smaller than the number used to represent four right angles, *i. e.*, in our measure positive numbers smaller than 64 (see fig. 75).

When we form sums or differences of the numbers which represent the given angles we may come both to positive numbers which are greater than 64, and to negative numbers. In such cases we shall always by subtraction or addition of 64 (or a multiple of 64) reduce to a positive number smaller than 64. This will always be allowed by the general reason that there is no difference between the direction represented by α and that represented by $\alpha \pm$ four right angles. This remark is of special importance in connection with *the difference of angle* $\beta-\alpha$, which represents the direction of the vector **B** *relatively to that of* **A**. When we agree always to represent this difference of angle by a positive number, it implies that we agree to count it as *produced by a rotation in positive direction from the vector* **A**, of which the angle α a ppears as subtractor *to the vector* **B**, of which the angle appears as minuend (see fig. 75).

These agreements must be remembered for the understanding of our charts, where the isogons, whether they represent absolute angles α, β . . . or differences of angle $\beta-\alpha$, are always numbered with positive numbers contained between 0 and 64.

Two vectors which cut each other under constant angle will have the same system of isogons, only with different numbers appearing on the isogons. The difference will be zero, if the two vectors have the same direction, 32 if they have the opposite direction, and 16 or 48 if they cut each other under right angle. Evidently two opposite directions will have equal right to be called normal to a given direction. We shall therefore agree to distinguish between these two directions by a rule of signs, namely this:

> *From a given direction we pass to that of its positive normal by a rotation of one right angle and to that of its negative normal by a rotation of three right angles in positive direction.*

It follows from this rule that when the vector **B** is directed along the positive normal to the vector **A**, the vector **A** will be directed along the negative normal to the vector **B**. Or in the notations (a): The vector

$$(B, \beta) = (B, \alpha+16)$$

is directed along the positive normal to the vector (A, α). But then

$$(A, \alpha) = (A, \beta-16) = (A, \beta+48)$$

will be directed along the negative normal to the vector (B, β).

156. Projections of a Vector; Scalar Product and Vector-Product.—Let a direction represented by the angle α be given everywhere in the field. We shall form the projection A_1 of a given vector (F, φ) on this direction. This projection will have the positive or the negative sign according as it points *in* or *against* the direction represented by the given angle α.

The projection is given by

$$(a) \qquad A_1 = F \cos (\varphi - \alpha)$$

We solve with respect to F, and to $\varphi - \alpha$

$$F = \frac{A_1}{\cos (\varphi - \alpha)} \qquad \varphi - \alpha = \text{arc} \cos \frac{A_1}{F}$$

We tabulate F as function of the variables A_1 and $\varphi - \alpha$ (first of tables K). In the same manner we should have tabulated $\varphi - \alpha$ as function of F and A_1. But as we deal here only with the general principles, and not with the tables for practical use, we shall give here and in several cases below only one table. The field of the projection A_1 can then be found in two operations. By graphical subtraction we form the field of the angle $\varphi - \alpha$. This field is placed upon that which represents the scalar value F of the given vector. Using the first table K, we derive from the curves $\varphi - \alpha =$ const. and $F =$ const., the field of the scalar A_1, proceeding as we have exemplified several times already for graphical operations with two variables. In this, as well as in several of the following tables, each tabulated number corresponds to different sets of arguments. The arguments on the left side and above belong together, and so do the arguments on the right side and below. In order to avoid mistakes it may be favorable for practical use to have two tables containing the same tabulated numbers, but each only with one set of arguments.

We can now form the projection of (F, φ) on the positive normal to that direction which is given by the angle α. For this projection we have

$$(b) \qquad A_2 = F \sin (\varphi - \alpha)$$

We solve this equation with respect to F

$$(b') \qquad F = \frac{A_2}{\sin (\varphi - \alpha)}$$

and tabulate F as function of A and $\varphi - \alpha$ (second table K). Thus, in order to find the field of this projection A_2, we first form the same auxiliary field $\varphi - \alpha$ as in the preceding case, place this field upon that which represents the intensity F of the given vector, and draw the curves $A_2 =$ const. by use of the second table K.

By the two tables K we can thus solve a vector F into orthogonal components A_1 and A_2. We thus have the way open to bring coordinate-methods into application when this should be desirable.

From expressions of the form (a) and (b) there is only one step to expressions of the form

$$AB \cos (\beta - \alpha) \quad \text{and} \quad AB \sin (\beta - \alpha)$$

i. e., to the formation of the complete scalar product or the complete vector-product

of two-dimensional vectors. Tables K in connection with a table for graphical multiplication will thus give the complete solution of the formation of these two products.

It is easier to explain the graphical procedures by formulæ and text than to illustrate them by text-figures, for the text-figures can not be placed upon each other on the illuminated drawing-board in order to make any two systems of curves visible at once as if they were drawn on the same sheet. This should be remembered when studying the example given in fig. 76, which illustrates the formation of the projection of the vector (F, φ) on the direction defined by the angle α. The chart A

TABLES K.—*Projections of a vector* (F, φ).

I. *Table for drawing the field of the projection* $A_1 = F \cos(\varphi - \alpha)$ (*F tabulated*).

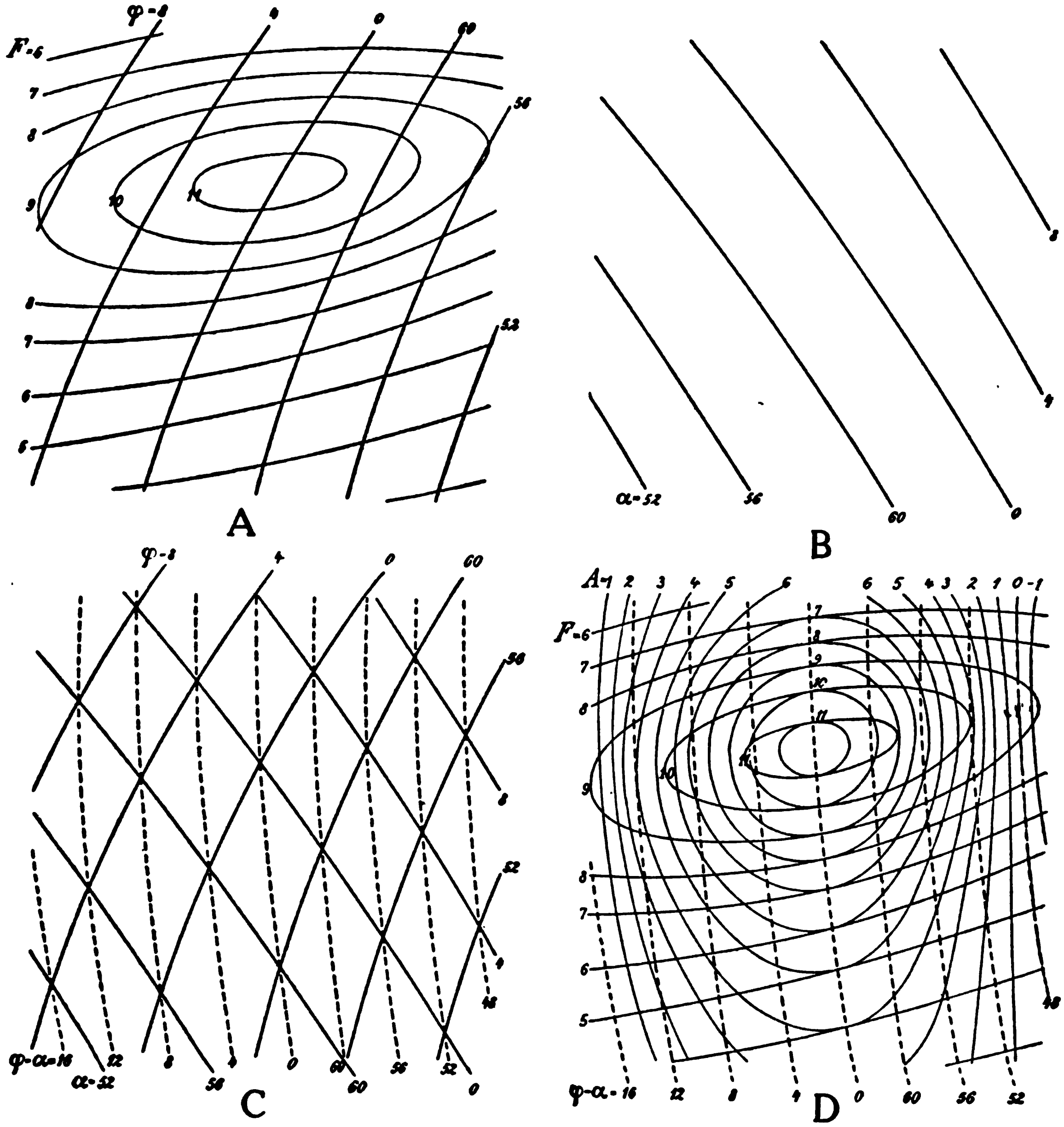
Projection A_1	Angle $(\varphi-\alpha)$.					
	0 64	4 60	8 56	12 52	16 48	
0	0	0	0	0	$\frac{0}{0}$	0
1	1	1.1	1.4	2.6	∞	−1
2	2	2.2	2.8	5.2	∞	−2
3	3	3.2	4.2	7.8	∞	−3
4	4	4.3	5.7	10.5	∞	−4
5	5	5.4	7.1	13.1	∞	−5
6	6	6.5	8.5	15.7	∞	−6
7	7	7.6	9.9	18.3	∞	−7
8	8	8.7	11.3	20.9	∞	−8
9	9	9.7	12.7	23.5	∞	−9
10	10	10.8	14.1	26.1	∞	−10
	32 32	28 36	24 40	20 44	16 48	A_1 Projection.
	Angle $(\varphi-\alpha)$					

II. *Table for drawing the field of the projection* $A_2 = F \sin(\varphi - \alpha)$ (*F tabulated*).

Projection A_2	Angle $(\varphi-\alpha)$.					
	0 32	4 28	8 24	12 20	16 16	
0	$\frac{0}{0}$	0	0	0	0	0
1	∞	2.6	1.4	1.1	1	−1
2	∞	5.2	2.8	2.2	2	−2
3	∞	7.8	4.2	3.2	3	−3
4	∞	10.5	5.7	4.3	4	−4
5	∞	13.1	7.1	5.4	5	−5
6	∞	15.7	8.5	6.5	6	−6
7	∞	18.3	9.9	7.6	7	−7
8	∞	20.9	11.3	8.7	8	−8
9	∞	23.5	12.7	9.7	9	−9
10	∞	26.1	14.1	10.8	10	−10
	0 32	60 36	56 40	52 44	48 48	A_2 Projection.
	Angle $(\varphi-\alpha)$					

represents the field of the given vector (F, φ), the chart B that of the given angle α. The sheets containing these two charts are placed upon each other, and on a third sheet we draw the curves for the difference of angle $\varphi - \alpha$ as illustrated by chart C. We then place the three sheets upon each other in reversed order, taking that which contains the field of α uppermost and draw upon it the curves $A =$ const. illustrated by chart D. On this sheet we have then obtained the chart E which contains the complete result. Thus, while fig. 76 contains five charts, only three sheets of paper have been used.

157. Addition of Vectors which are Normal to Each Other.—In two-dimensional vector-algebra the vector-product has lost its character of a proper vector. It can be treated as a scalar, and is therefore easier to form than the vector-sum, which always remains a true vector. In two-dimensional vector-algebra the vector-addition is therefore the only typical vector-operation, and as a rule an operation of more complicated nature than the formation of the products.



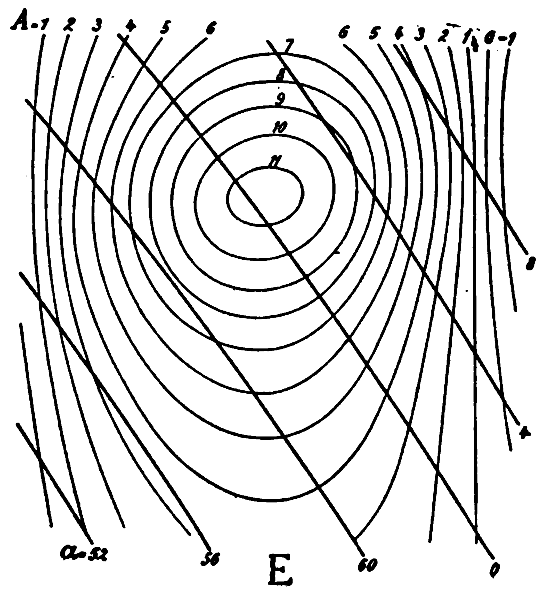


FIG. 76.—Projection of a vector.

A. Field of given vector (F, φ) represented by intensity-curves $F = 6, 7, 8, \ldots$ (fine) and isogons $\varphi = 8, 4. 0, 60 \ldots$ (heavy).

B. Isogons of angle $a = 52, 56, 60, \ldots$

C. Isogons of angle a of the angle φ and of the difference $\varphi - a = 16, 12, 8, \ldots$ (stippled).

D. Isogons $\varphi - a$, intensity-curves F, and intensity-curves $A = 1, 2, 3, \ldots$ of projection derived by first of tables K.

E. Field of projection (A, a) represented by intensity-curves A and isogons a.

TABLES L.—*Graphical addition of mutually normal vectors.*

I. *Table for drawing the intensity field* $F = \sqrt{A^2+B^2}$ *(B or A tabulated).*

Intensity of vector-sum. F	Intensity of first vector-addend A.											
	0	1	2	3	4	5	6	7	8	9	10	
0	0											0
1	1	0										1
2	2	1.7	0									2
3	3	2.8	2.2	0								3
4	4	3.9	3.5	2.6	0							4
5	5	4.9	4.6	4.0	3.0	0						5
6	6	5.9	5.7	5.2	4.5	3.3	0					6
7	7	6.9	6.7	6.3	5.7	4.9	3.6	0				7
8	8	7.9	7.7	7.4	6.9	6.2	5.3	3.9	0			8
9	9	8.9	8.8	8.5	8.1	7.5	6.7	5.7	4.1	0		9
10	10	9.9	9.8	9.5	9.2	8.7	8.0	7.1	6.0	4.4	0	10
	0	1	2	3	4	5	6	7	8	9	10	F Intensity of vector-sum.
	Intensity of second vector-addend B.											

II. *Table for drawing the field of the angle* $\varphi-\alpha$ *between the vector-sum F and the vector-addend A (B or A tabulated).*

Angle $(\varphi-\alpha)$		Intensity of first vector-addend A.												
$\beta-\alpha=16$	$\beta-\alpha=48$	0	1	2	3	4	5	6	7	8	9	10		
0	64 or 0	0	0	0	0	0	0	0	0	0	0	0	16	48
4	60	0	0.4	0.8	1.2	1.7	2.1	2.5	2.9	3.3	3.7	4.1	12	52
8	56	0	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0	8	56
12	52	0	2.4	4.8	7.2	9.7	12.1	14.5	16.9	19.3	21.7	24.1	4	60
16	48	$\frac{0}{0}$	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	0	64 or 0
		0	1	2	3	4	5	6	7	8	9	10	$\beta-\alpha=16$	$\beta-\alpha=48$
		Intensity of second vector-addend B.											Angle $(\varphi-\alpha)$	

We shall first treat the case in which the two given vectors **A** and **B** are normal to each other. The problem of their addition may then be considered as inverse to the problem of solution into rectangular components treated in the preceding section. It will be precisely inverse if the two vectors **A** and **B** are given in the form used for components along orthogonal coordinate-curves, *i. e.*, by equiscalar curves for positive and negative values of the intensity. On the other hand, it will take a slightly changed form if the vector is given in the ordinary way by isogons and intensity curves for an always positive intensity. We shall treat this case only. It will easily be seen that in the other case we can use the same tables, only with a

somewhat changed arrangement as to the sign of the tabulated numbers and the arguments. For the scalar value of the vector (F, φ) which is the resultant of the vectors (A, α) and (B, β) we have

(a) $$F^2 = A^2 + B^2$$

We solve this equation with respect to one of the given quantities A or B,

(a′) $$B = \sqrt{F^2 - A^2}, \text{ or } A = \sqrt{F^2 - B^2}$$

Both formulæ lead to the same table, the first of tables L, where, according to circumstances we can consider F and A or F and B as arguments. By this table we can thus derive the intensity-curves for the vector-sum from the intensity-curves of the two orthogonal vector-addends.

In order to form the isogons of (F, φ) we have to remember that the vector (B, β) in some regions of the field may be directed along the positive and in others along the negative normal to (A, α). In the two cases we shall have respectively

$$\beta - \alpha = 16 \text{ and } \beta - \alpha = 48$$

with corresponding values of the angle $\varphi - \alpha$

$$\varphi - \alpha < 16 \text{ and } \varphi - \alpha > 48$$

The rectangular triangle will give for the determination of this angle in the two cases respectively

(b) $$tg\ (\varphi - \alpha) = \frac{B}{A} \text{ and } tg\ (\varphi - \alpha) = -\frac{B}{A}$$

We solve these equations with respect to one of the given quantities A or B, thus

(b′) $$\begin{aligned} B &= A \operatorname{tg} (\varphi - \alpha) \quad \text{and} \quad & B &= -A \operatorname{tg} (\varphi - \alpha) \\ A &= B \operatorname{cotg} (\varphi - \alpha) & A &= -B \operatorname{cotg} (\varphi - \alpha) \end{aligned}$$

By suitable change of arguments all formulæ can be represented by one table, the second of tables L. This table allows us to derive the field of the angle $\varphi - \alpha$ from the fields of the two tensors A and B.

When the field of the angle $\varphi - \alpha$ is found, we find by the graphical addition

(c) $$\varphi = (\varphi - \alpha) + \alpha$$

the field of the angle φ which represents the direction of the vector-sum.

The illustration of the procedure by text-figures would seem complicated, but when the illuminated drawing-board is used, only four sheets of paper are required: two contain the fields of the given vectors; a third is used for the field of the auxiliary quantity $\varphi - \alpha$; on the fourth we draw directly the final curves giving the fields of F and of φ.

158. Addition of Any Vectors.—When the two given vectors (A, α) and (B, β) cut each other under a variable angle, the operation of determining their vector-sum (F, φ) will depend upon four variables, A, α, B, β. But the complex operation can be decomposed into the following series of operations, each involving the use of two variables.

(1) By graphical subtraction we form the auxiliary field of the scalar $\beta-\alpha$ which represents the angle between the two given vectors.

(2) By graphical division (section 151) we form the auxiliary field representing the ratio $\frac{B}{A}$ of the numerical values of the two given vectors.

(3) By the elementary properties of the triangle with the sides A, B, and F we get the following relation connecting the angle $\varphi-\alpha$ with the known angle $\beta-\alpha$ and the known ratio $\frac{B}{A}$

$$(a) \qquad \left(\frac{B}{A}-1\right)\operatorname{tg}\frac{\beta-\alpha}{2} = \left(\frac{B}{A}+1\right)\operatorname{tg}\left(\varphi-\alpha-\frac{\beta-\alpha}{2}\right)$$

We solve this equation with respect to $\frac{B}{A}$ and tabulate this quantity as function of the two angles $\beta-\alpha$ and $\varphi-\alpha$. Using this table, the first of tables M, we can derive the field of the angle $\varphi-\alpha$ from the fields of the two auxiliary quantities $\beta-\alpha$ and $\frac{B}{A}$.

(4) By the properties of the same triangle we find the following relation which connects the ratio $\frac{F}{A}$ with the ratio $\frac{B}{A}$ and the angle $\beta-\alpha$,

$$(b) \qquad \left(\frac{F}{A}\right)^2 = 1+\left(\frac{B}{A}\right)^2+2\frac{B}{A}\cos(\beta-\alpha)$$

We solve this equation with respect to $\frac{B}{A}$ and tabulate this quantity with the ratio $\frac{F}{A}$ and the angle $\beta-\alpha$ as arguments. This gives the second of tables M. Using this table we can derive the field of the ratio $\frac{F}{A}$ from the fields of the two auxiliary quantities $\beta-\alpha$ and $\frac{B}{A}$. When two numbers are given in the same place in the table, the curve $\frac{F}{A}$ = const. has two points of intersection with the curve $\beta-\alpha$ = const. Both will have to be used.

(5) By graphical multiplication (section 150) we derive the field of the intensity F of the required vector from the fields of the ratio $\frac{F}{A}$ and of the intensity A of the given vector.

(6) By graphical addition we derive the field of the angle φ of the required vector from the fields of the angles $\varphi-\alpha$ and α.

It should be emphasized that as soon as we have drawn the first two systems of auxiliary curves (1) and (2), we know the situation of all zero-points of the field.

Every singular point of a vector is a zero-point for its absolute value. The resultant can be zero only in points where the two given vectors have equal magnitude and opposite direction. Now the two given vectors have equal magnitude in the points of the curve $\frac{B}{A} = 1$, and opposite direction in the points of the curve $\beta-\alpha = 32$.

Hence it follows:

The singular points in the field of the vector-sum are the points of intersection of the curves

(c) $$\beta - \alpha = 32 \quad \text{and} \quad \frac{B}{A} = 1$$

On account of the ample information which the singular points give regarding the field, it will be important to draw with as great care as possible these two curves. In return much labor can be saved in other parts of the field, where few additional data are required besides those involved in the knowledge of the situation of the singular points.

TABLES M.—*Graphical addition of any vectors.*

I. *Table for drawing the field of the angle between the vector-sum and the vector-addend A.*

Angle $\varphi-\alpha$	Angle $(\beta-\alpha)$.									
	0	4	8	12	16	20	24	28	32	
0	$\frac{0}{0}$	0	0	0	0	0	0	0	<1	0
4		∞	1.00	0.54	0.41	0.38	0.41	0.54	1.00	60
8			∞	1.85	1.00	0.76	0.71	0.76	1.00	56
12				∞	2.4	1.30	1.00	0.92	1.00	52
16					∞	2.6	1.41	1.08	1.00	48
20						∞	2.4	1.30	1.00	44
24							∞	1.85	1.00	40
28								∞	1.00	36
32									>1	32
	64	60	56	52	48	44	40	36	32	$\varphi-\alpha$ Angle
	Angle $(\beta-\alpha)$									

II. *Table for drawing field of ratio of intensity of the vector-sum to that of the vector-addend A.*

Ratio $\frac{F}{A}$	Angle $(\beta-\alpha)$								
	0	4	8	12	16	20	24	28	32
	64 or 0	60	56	52	48	44	40	36	32
0									1.00
0.5								0.60	0.50
0.5								1.25	1.50
1.0						0.00	0.00	0.00	0.00
1.0					0	0.77	1.42	1.85	2.00
1.5	0.50	0.53	0.62	0.80	1.12	1.57	2.04	2.38	2.50
2.0	1.00	1.04	1.16	1.39	1.73	2.16	2.58	2.89	3.00
3.0	2.0	2.1	2.2	2.4	2.8	3.2	3.6	3.9	4.0
4.0	3.0	3.1	3.2	3.5	3.9	4.3	4.6	4.9	5.0
5.0	4.0	4.1	4.2	4.5	4.9	5.3	5.6	5.9	6.0
6.0	5.0	5.1	5.2	5.6	5.9	6.3	6.7	6.9	7.0
7.0	6.0	6.1	6.3	6.6	6.9	7.3	7.7	7.9	8.0
8.0	7.0	7.1	7.3	7.6	8.0	8.3	8.7	8.9	9.0
9.0	8.0	8.1	8.3	8.6	9.0	9.3	9.7	9.9	10.0
10.0	9.0	9.1	9.3	9.6	10.0	10.3	10.7	10.9	11.0

159. Easily Accessible Data Regarding the Field of the Vector-Sum.—Just as we can find the singular points, we can easily find a series of further data regarding the field of the vector-sum. It will save much labor to use these data as completely as possible.

The vector-addition will be performed according to the simplest law, that of scalar addition or subtraction, in all points where the two given vectors have either the same or opposite directions, *i.e.*, in the points of the curves

(*a*) $$\beta - \alpha = 0$$
(*b*) $$\beta - \alpha = 32$$

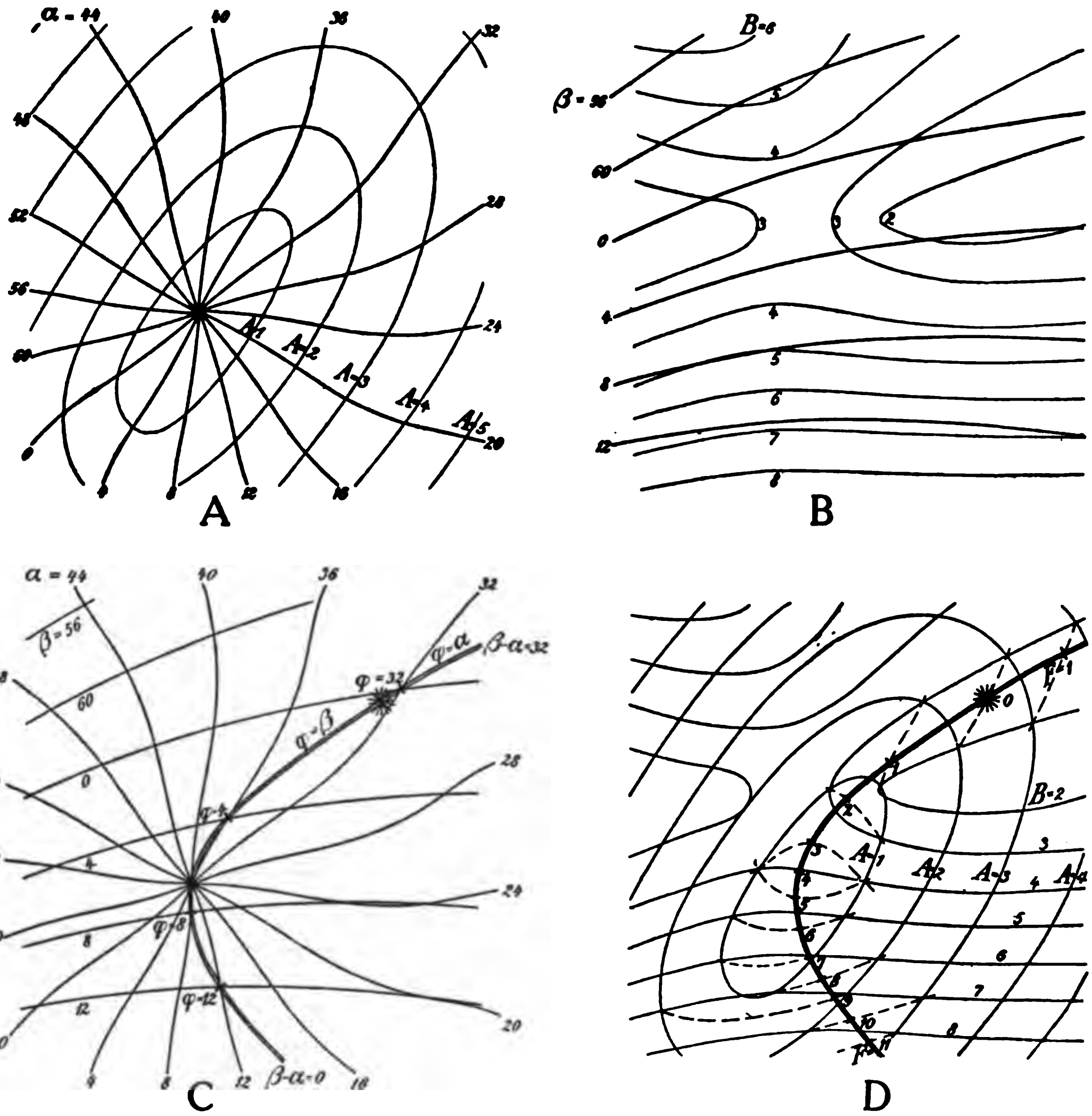


FIG. 77.—Easily accessible data on the field of the vector-sum.

A. First given vector (A, α).
B. Second given vector (B, β).
C. Curves $\beta - \alpha = 0$ and $\beta - \alpha = 32$, and their points of intersection with the required curves $\varphi = 12, 8, 4$.
D. Curves $\beta - \alpha = 0$ and $\beta - \alpha = 32$ and their points of intersection with the required curves $F = 11, 10, 9, \ldots$

In all points of the first curve the vector-sum **F** will have the same direction as both **A** and **B**, and a numerical value equal to their scalar sum. In all points of the second curve the vector-sum **F** will have the same direction as the greater of the vectors **A** and **B**, and a numerical value equal to their scalar difference. We can therefore

with the greatest ease find all data regarding direction and intensity of the vector-sum in all points of the two curves (a) and (b).

The curves (a) and (b) belong to the first set of auxiliary curves drawn for the determination of the vector-sum, section 158 (1). While we draw the curve (a) we can mark on it the points where it will be cut by the required curves $\varphi = 0, 1, 2, \ldots$ for these will be the same points as those which serve for the determination of the curve $\beta - \alpha = 0$ itself, namely, the points of intersection of the curve $\alpha = 0$ with $\beta = 0$, of the curve $\alpha = 1$ with $\beta = 1$, of the curve $\alpha = 2$ with $\beta = 2$, and so on (fig. 77 C). In order to find the points where the same curve is cut by the required intensity-curves $F = 0, 1, 2, \ldots$, we have simply to draw the short parts of the curves $A + B = 0, 1, 2, \ldots$ which cut the curve (a) (see fig. 77 D).

In the same manner, while we draw the curve (b), we can mark on it the points where it will be cut by the curves $\varphi = 0, 1, 2, \ldots$; for these points will again be the same as those points of intersection of the given curves $\alpha =$ const. and $\beta =$ const. which serve to determine the curve (b). We have to remark that the integer values of φ which should be noted at these points will be those of α when $A > B$ and those of β when $B > A$. In order to find the points where the same curve is cut by the intensity-curves $F = 0, 1, 2, \ldots$, we have to draw the parts of the curves $|A - B| = 0, 1, 2, \ldots$, which cut the curve ($b$) (see fig. 77 D). Evidently the intersection of the curve $A - B = 0$ with the curve (b) gives the singular points of the field of the vector-sum. It should be observed that the curve $B - A = 0$ is identical with the curve $\frac{B}{A} = 1$, which we have used already in the preceding section for the determination of the singular points. These points will divide the curve (b) into distinct branches. As we pass a singular point the value of the angle φ will change suddenly from $\varphi = \alpha$ to $\varphi = \beta$, or vice versa.

Thus the investigation of the two curves (a) and (b) gives with great ease both the situation of the singular points in the field of the vector-sum and in addition a great number of points through which different curves representing the field of the vector-sum shall pass. These data can be utilized in different ways, according to the method otherwise used for finding the field of the vector-sum. If the method given in the preceding section be retained, it will be important to remark that the curves (a) and (b) will turn up again as the curves

(c) $$\varphi - \alpha = 0$$
(d) $$\varphi - \alpha = 32$$

in the auxiliary field of the angle $\varphi - \alpha$, which is found by the operation (3) of the preceding section. The curve (c) will correspond to the curve (a) and certain parts of the curve (b), the change in the correspondence taking place at the singular points.

160. Graphical Tables for Vector-Addition.—Our first solution of the problem of graphical vector-addition depended upon the decomposition of the general problem with four variables into six partial problems each with two variables. But if we use the method which we have developed in section 152 for three varia-

bles, we can reduce to a smaller number of partial problems. The operations with three variables which we shall have then to perform will join themselves directly to those of the preceding section.

After we have drawn the curves

$$(a) \qquad \beta - \alpha = \text{const.}$$

we can pass directly to the determination of the angle $\varphi - \alpha$ and of the intensity F of the resultant by the formulæ

$$(b) \qquad \operatorname{tg}(\varphi - \alpha) = \frac{B \sin(\beta - \alpha)}{A + B \cos(\beta - \alpha)}$$

$$(c) \qquad F^2 = A^2 + B^2 + 2AB \cos(\beta - \alpha)$$

In each of these formulæ we can give $\beta - \alpha$ a certain constant value and by the principles of section 152 construct a graphical table by which we can find the points in which this particular curve $\beta - \alpha = \text{const.}$ is cut by the curves for integer values of $\varphi - \alpha$ and of F. We then set off A and B as abscissa and ordinate of a rectangular system of coordinates, and draw in the one case the curves $\varphi - \alpha = \text{const.}$, in the second the curves $F = \text{const.}$ in this system of coordinates. It will be seen at once that the first curves are simply straight lines through the origin of the coordinates, the second ellipsæ with the origin of the coordinates as center and with the axes forming the angle 8 (45°) with the axes of coordinates. It will be convenient to draw both systems of curves on the same diagram. Then we can read off simultaneously the situation of the required points for integer values both of $\varphi - \alpha$ and of F.

In fig. 78 we have drawn these diagrams for the values $\beta - \alpha = 4, 8, 12, 16, 20, 24$. The radial lines $\varphi - \alpha = \text{const.}$ are drawn in these diagrams for the interval 4. Thus on the first diagram $\beta - \alpha = 4$ we have only two lines $\varphi - \alpha = \text{const.}$, namely, the two axes of coordinates. On the following we have 3, 4, 5, 6, and 7 of them respectively. The ellipsæ are drawn for unit intervals of the intensity F. The ratio of the axes changes as we pass from the one diagram to the other. In the case of $\beta - \alpha = 16$, *i.e.*, at the curve where the vectors cut each other under right angle, the two axes are equal to each other and the curves are circles. It will easily be seen that the same diagrams may be used for the values 60, 56, 52, 48, 44, 40 of $\beta - \alpha$, taken in connection with the values of $\varphi - \alpha$, which are written in brackets on the diagrams.

By use of these diagrams, including the first of figs. 101, p. 127, we can then find the points in which the curves for integer values of $\varphi - \alpha$ and of F cut 14 isogons $\beta - \alpha = \text{const.}$ The points of intersection with the fifteenth and the sixteenth, viz, $\beta - \alpha = 0$ and $\beta - \alpha = 32$, have been found already by the simpler method of the preceding section.

A great advantage of this method is that two draftsmen can cooperate. One has before him a chart containing the three sets of curves $A = \text{const.}$, $B = \text{const.}$, and $\beta - \alpha = \text{const.}$ They may be copied on one paper, or they may be drawn on three different papers which are placed upon each other on the illuminated drawing-board. The other has the graphical table fig. 78 and a transparent paper placed upon it.

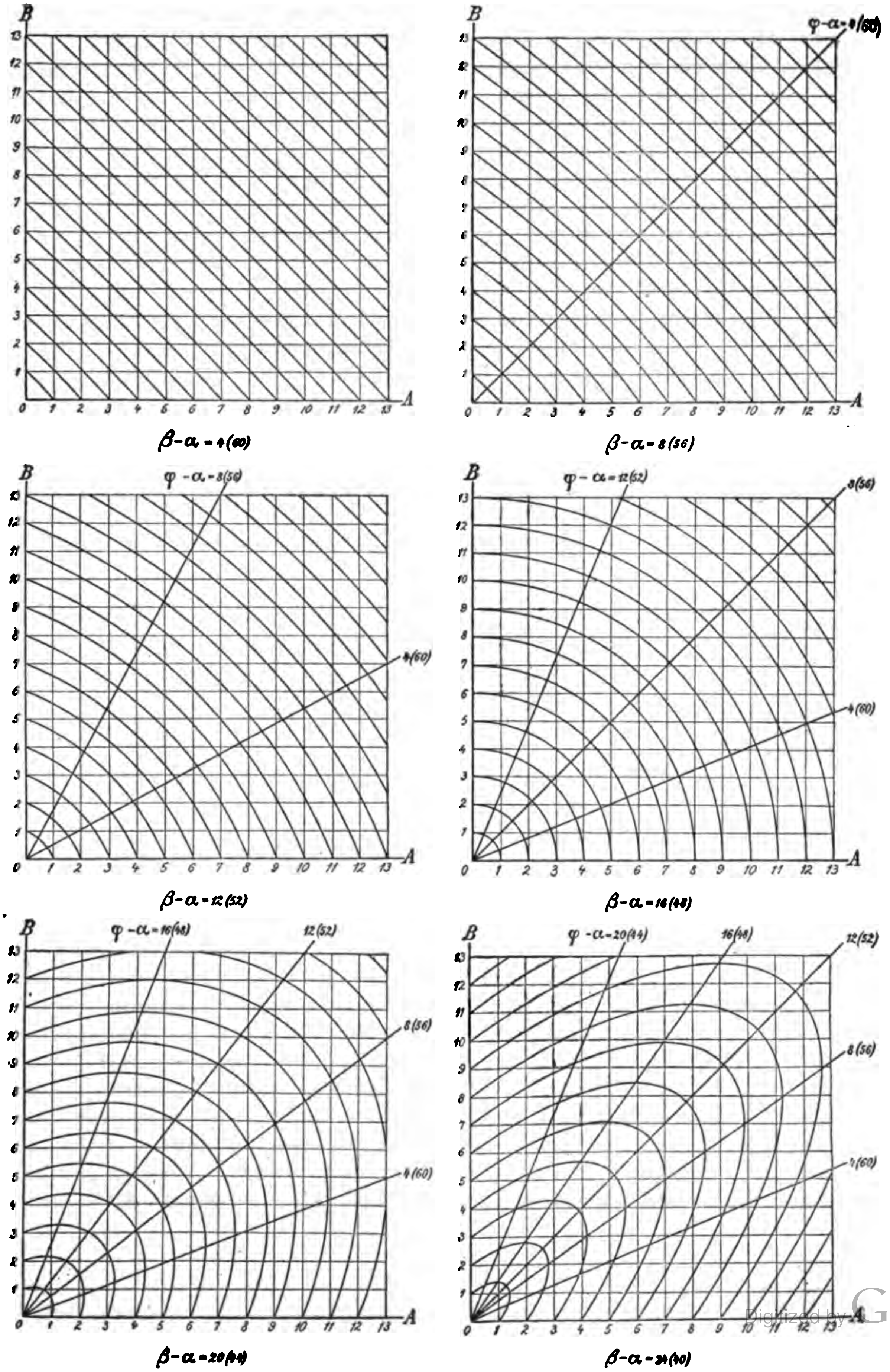


FIG. 78.—Graphical tables for vector-addition.

Let it be required, for instance, to determine the points in which the curve $\beta-\alpha=20$ is cut by the curves $\varphi-\alpha=0, 4, 8, 12, \ldots$ and by the curves $F=1, 2, 3, 4, \ldots$ The first draftsman then observes the connected values of A and of B along the curve $\beta-\alpha=20$, and dictates that it cuts the curve $A=1$ in the point where $B=B_1$, the curve $A=2$ in the point where $B=B_2$, etc. The second draftsman then draws point by point the corresponding curve on the transparent sheet placed upon fig. 78. Then the second draftsman follows the course of the curve which he has drawn, and dictates to the first that it cuts the curve $F=1$ at the point where $A=A_1$, the curve $F=2$ at the point where $A=A_2 \ldots$, the curve $\varphi-\alpha=0$ at the point where $A=A'_0$. the curve $\varphi-\alpha=4$ at the point where $A=A'_4, \ldots$ The first draftsman then marks these points on the curve $\beta-\alpha=20$ on his chart, using different kinds of marks for the curves $F=$ const. and $\varphi-\alpha=$ const., and adding the numerical values of F and $\varphi-\alpha$. When this is repeated for a sufficient number of curves $\beta-\alpha=$ const. we shall get a complete set of points determining the course of the curves $F=$ const. and $\varphi-\alpha=$ const.

From the set of curves $\varphi-\alpha=$ const. we finally pass, by the graphical addition $(\varphi-\alpha)+\alpha=\varphi$, to the curves representing the required angle φ.

When we compare with the method of section 158, we see that the use of the graphical tables replaces the performance of the separate graphical operations (2), (3), (4), (5). Only the simple graphical subtraction (1) and the graphical addition (6) are retained.

161. Complete Resultantometer.—While the method of section 158 required the drawing of four auxiliary systems of curves, besides the fifth and sixth, which represent the result, we succeed by using the graphical tables in arriving at the result by drawing only two auxiliary systems of curves. By introducing a still more complete auxiliary, a complete machine for vector-addition, we can completely avoid the drawing of auxiliary systems of curves.

Instruments for adding vectors can be constructed in various ways, each having an advantage according to the special form in which the problem presents itself. Fig. 79 shows a convenient construction for our purposes. We draw parallel and equidistant lines on two circular transparent sheets and concentric circles on one of them. The sheets are laid upon each other, so that the upper is free to slide inside the divided brass-ring C, while the lower is mounted in a brass-ring which can slide outside the ring C. This ring contains the divisions 0 to 63, which represent the angles. When the instrument is to be used on our charts in conical projection, the ring C is attached to the rule R, which passes through the point of convergence of the meridians. (Compare the integration-machine of fig. 62.) The divisions of the circle C will then always show the true directions relatively to the meridians on the chart. For the practical use of the instrument it will finally be useful to have two screws by which we can attach either of the divided sheets rigidly to the ring C and thus give the lines of the fixed sheet an invariable direction relatively to the meridians of the chart. The two sheets are perforated at the center, in order to make it possible to set marks on the paper underneath by use of a pin or a sharp pencil. All lines are

engraved on the upper side of the lower sheet and on the under side of the upper sheet. When using the instrument it will be best to have illumination from below. Otherwise the pictures seen will be blurred by the shadows which the lines will throw on the paper.

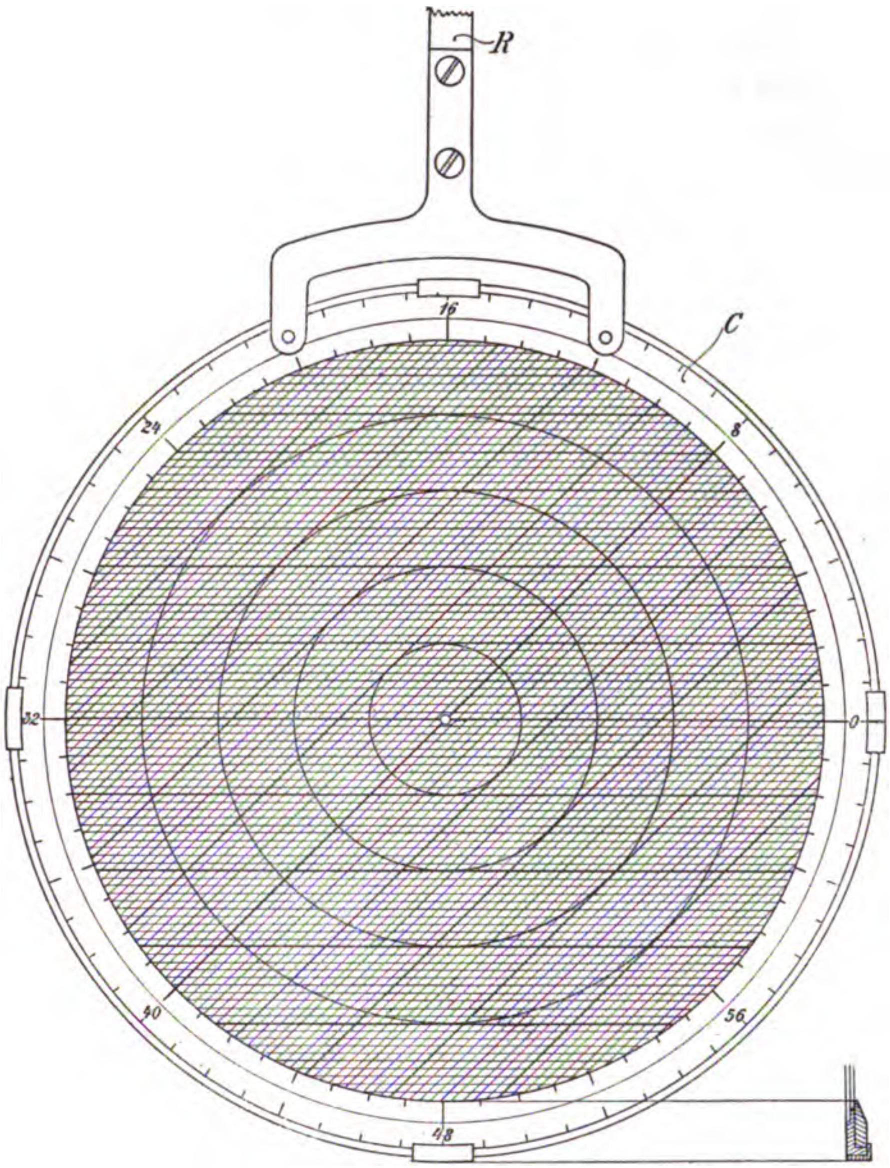


FIG. 79.—Resultantometer.

When one of the divided sheets is made to rotate relatively to the other, the two sets of parallel lines produce parallelograms of all possible shapes. The intersection of the lines gives at the same time equidistant divisions on each line, which can be used for measuring the lengths which shall represent the scalar values of the given vectors. In this manner we use a variable unit length, which varies with the angle between the lines. This is a great advantage, for we get automatically the construction performed on a large scale in the difficult case in which the given vectors

are nearly equal and oppositely directed. The direction of the resultant is read off on the divided circle C. In order to read off the intensity of the resultant on the same scale as that used for the components, we follow the circles from the point at the end of the resultant to one of the central lines of the one divided sheet.

The discontinuous use of the instrument will be understood at once. If the directions of the given vectors are represented by vector-lines, the two divided sheets are adjusted so as to be tangential to one line of each set. If the directions are given by isogons, the adjustment of the sheets is made by use of the divided circle C. In this case it will not be necessary to place the instrument on the chart. Two workers can cooperate. One manages the instrument, while the other reads off from the chart the given data and introduces the results on it.

Continuous use of the instrument will also be possible. We can then go along an isogonal curve, having the one disk fixed in the angle represented by the isogon, while the other is turned according to the value of the angle represented by the other set of isogons. The intensities of the two vectors are observed, and thus by short steps we can follow the variations of the angle and the intensity of the resultant and mark the points where integer values occur. But this work will require keen attention.

CHAPTER IX.

GRAPHICAL DIFFERENTIATION AND INTEGRATION.

162. Different Forms of the Problems.—We shall meet with problems of graphical differentiation in a variety of forms, each requiring the development of special methods and auxiliaries. The problems will take different forms according as *space* or *time* derivations should be performed. The pure space-derivations will depend upon measurements performed upon a chart which represents the given field at a given epoch. The time derivations will involve a comparative investigation of *two* charts which represent the fields of the same quantity at two different epochs. We shall consider first the space-derivations and afterwards the time-derivations.

The space-derivations will present themselves in different forms, requiring different methods and auxiliaries according as they depend upon the measurement of lengths or of angles. We shall consider first the angular or directional and then the linear differentiations. To each problem of the differentiation will correspond a problem of integration which in the elementary cases will cause no difficulty as soon as the problem of differentiation is solved.

163. Directional Differentiation and Integration.—Let a system of curves s be given; by their tangents they define a system of directions. It is required to find the angle φ which gives the direction of the tangents, *i. e.*, we shall draw the isogonal curves which represent the field of this angle. Evidently this is a problem of differentiation which is inverse to the problem of integration, consisting in the drawing of the vector-lines to a given system of isogons. This drawing of the isogons to a given system of curves can be performed with a certain degree of precision by eye-measure, but a simple auxiliary instrument will be of great use. A transparent circular sheet (fig. 80) can slide in a ring, which has the divisions 0 to 63 or a certain number of these divisions. On the sheet is drawn a diameter and a set of lines parallel to it. Millimetric distance between them will in most cases be convenient. The ring is guided so that it has invariable orientation relatively to the system of coordinates. Thus if cartesian coordinates are used, the ring is guided so that it can perform any motion of translation without rotation. In the case of our charts in conical projection the ring is attached to the rule R, which always passes through the point of convergence of the meridians. (Compare fig. 62.) The sheet has a small perforation at the center, which allows us to mark the points where the desired values of the angle are found.

The sheet is guided in such a way that its center (the hole) follows one of the given curves. During the displacements it is turned so that the diameter remains tangent to the curve. The adjustment to tangency will be very much assisted by the lines which are parallel to the diameter. Whenever the diameter points to

one of the integer divisions on the ring we make a mark on the curve through the hole. In this manner the disk is guided along the given curves, and marks are made where the required isogonal curves should intersect them. Afterwards these isogonal curves can be drawn continuously. If they are made to pass precisely through the points marked they will always show oscillations in their course, due to the unavoidable errors accompanying the drawing of the given curves and the use of the differentiating instrument. But these irregularities are easily smoothed out on the final drawing of the curves.

It will be important to remember that the curves which we obtain by this instrument can be numbered so as to be the isogons of the curves s themselves, or so as to be the isogons of the curves which are normal to the curves s. We pass

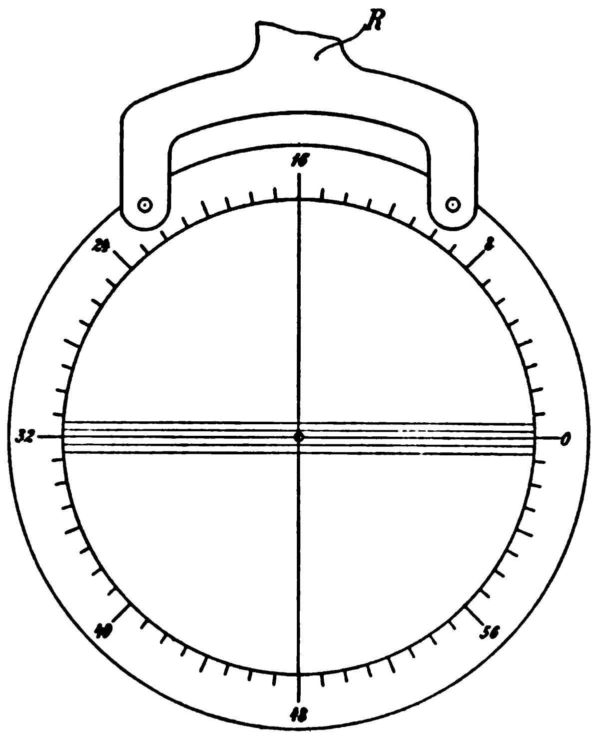


FIG. 80.—Divided sheet for directional differentiation.

from the isogons of the curves s to those of their positive normal curves by an addition of 16, to those of their negative normal curves by an addition of 48 to the numbers which the isogons have when they represent the curves s.

We have treated already (Chapter VII) the problem of integration which is inverse to the directional differentiation. Evidently the sheet of fig. 80, by which we perform the differentiations, may also be used to assist the integrations; and the integration-machine of fig. 62 or 63 can be considered as intrinsically the same instrument as that of fig. 80, only provided with special devices for facilitating the practical work connected with the integration.

164. Linear Differentiation and Integration.—Let the scalar a have a definite value at every point of a line s; *i. e.*, let a be a function of the length of arc s

$$a = a(s) \tag{a}$$

We represent this function by marking the points where it has certain integer values, $a_0, a_1, a_2, \ldots a_n, a_{n+1}, \ldots$ The expression "integer" must be taken in the same generalized sense of the word as before (section 147). The differences between the values of a in consecutive points will then also be expressed by "integer" numbers, and they must be small enough to be considered as differentials, $da = a_{n+1} - a_n$. The distance between the points will be the corresponding differentials of line ds, and the problem of differentiation will consist in forming the values of the quotient

$$(b) \qquad \varphi(s) = \frac{da}{ds}$$

at the different points of the line s.

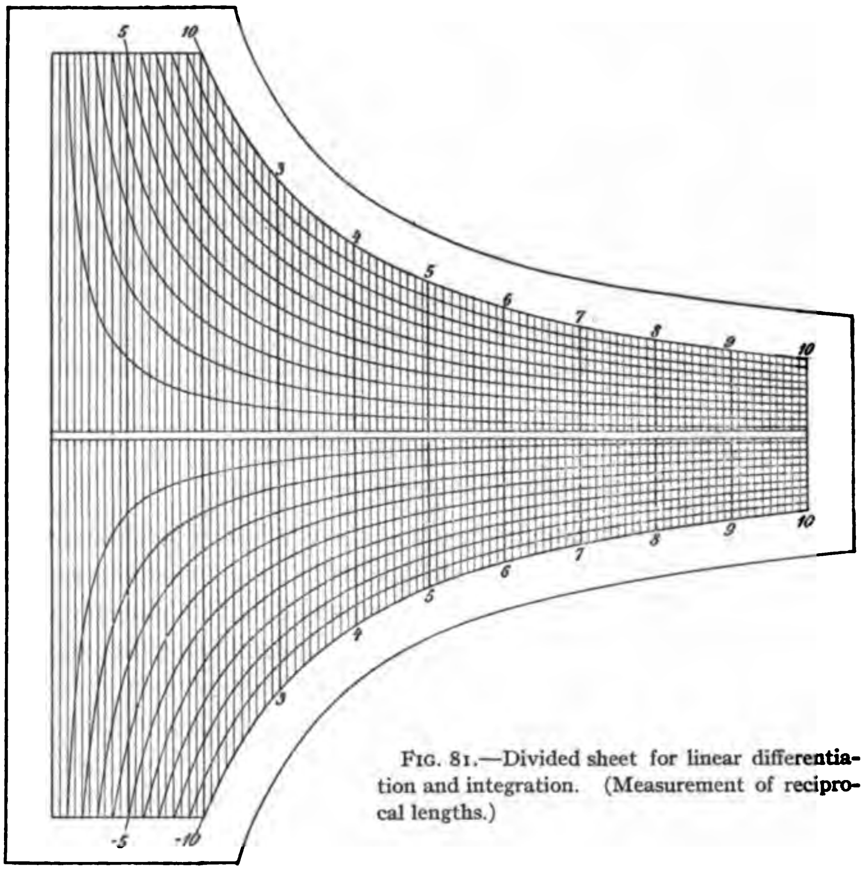


FIG. 81.—Divided sheet for linear differentiation and integration. (Measurement of reciprocal lengths.)

In order to construct a convenient auxiliary for the formation of the value of φ in one operation, we solve equation (b) with respect to da

$$(c) \qquad da = \varphi ds$$

We measure off $\varphi = x$ along the axis of abscissæ and $ds = y$ along the axis of ordinates of a rectangular system of coordinates. To each positive or negative integer value of da, viz, $da = \ldots -2, -1, 0, 1, 2, \ldots$ will then correspond an equilateral hyperbola $xy =$ const. The diagram of fig. 81 contains these curves together with a number of ordinates, one for each millimeter. Now let a value of the differential da be given, say $da = 4$. The abscissæ of the hyperbola $da = 4$ then gives the values

of φ corresponding to the line-element ds measured off as ordinates. If the length of this element is given, we can read off on the axis of abscissæ the corresponding value of the ratio $\varphi = \dfrac{da}{ds}$. Instead of measuring the length ds between the hyperbola 4 and the axis of abscissæ, we can also measure it between the two symmetric hyperbolæ $da = +2$ and $da = -2$. This will as a rule be preferable.

For practical use we engrave the diagram on the under side of a transparent sheet of celluloid, and cut a narrow slit in this sheet along the axis of abscissæ. The slit should just be broad enough to make it possible to make marks with a sharp pencil on the paper below the sheet. The sheet is placed so that the line-element ds is parallel to the ordinates of the sheet. In the case $da = 4$ it will have one end-point on each of the two hyperbolæ $da = +2$ and $da = -2$. The reading on the axis of the abscissa gives the value $\varphi = \dfrac{da}{ds}$, which the derivative has in the central point of the line-element ds. This point can be marked through the slit. It will be clear how different hyperbolæ should be used according to the occurring values of the differential da. The procedure is illustrated by the upper line of fig. 82, where the points for integer values of the function a are marked on the upper side of the line, while the values determined for the derivative are noted on the under side.

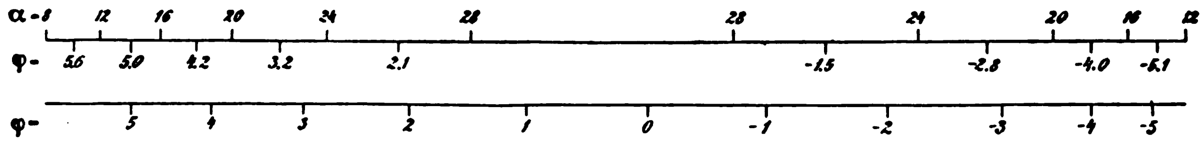


FIG. 82.—Linear differentiation or integration.

When the line-elements ds are short, a small error in the placement of the points where the given function has integer values will cause great errors in the values of the derivative. It will then be an excellent method of reducing these errors to measure two or more elements simultaneously. Thus if the points for all integer values $a = 1, 2, 3, \ldots$ are given, we measure the corresponding elements two by two between the hyperbolæ $+1$ and -1; or we can measure them four by four between the hyperbolæ $+2$ and -2, and so on.

As it is seen, the direct use of the sheet gives the derivative at points where it has all sorts of fractional values. But it will be easy afterwards by interpolation to find the points where the derivative has certain integer values. In the example of fig. 82 these points are marked on the lower line.

We can now treat the inverse problem, the linear integration. Then let the function $\varphi(s)$ be given. The problem is to determine any function $a(s)$, which is in the relation to the given function φ which is defined by equation (b) or (c). Evidently this can be done by the same divided sheet. For the sheet at once gives those lengths ds to which integer increases da will correspond. The process of integration must begin at a certain initial point $s = s_0$ and we presume that at this the required function has a given value $a = a_0$.

Now let the value of the given function in the region of this initial point be $\varphi = \varphi_0$. In order to find the point where a has the value a_0+4 we set off from the

initial point a length ds equal to the ordinate which the hyperbola $da = 4$ has for the value φ_0 of the abscissa. Using the value $\varphi = \varphi_1$ which φ has in the region of this new point, we measure off in the same manner the length ds, which leads to the point where a has the value a_0+8, and so on. Inasmuch as a_0 is integer, we find in this manner points for integer values of the function a. If we wish to proceed by other steps da, we use other hyperbolæ.

The marking off of the successive points can be made without removing the sheet from the paper. Thus in the case of $da = 4$ the sheet is placed with the hyperbola 4 on the point from which the length ds is to be measured. The new point can then be marked through the slit in the sheet.

We have spoken above of the value which the given function φ has in the "region" of the point from which the length ds is to be measured. This "region" will have a maximal extent equal to the length of the line-element ds. The use of one value or another of φ from this region will give no appreciable difference in the lengths ds obtained, if these lengths are sufficiently short; but the greater the lengths ds used, the more important it will be not to choose an arbitrary value of φ in this region, *but the average value of φ along the element ds.* As the approximate length of ds is seen at once, it will cause no difficulty to find a sufficiently approximate value of this average value of φ, and to use it for the final determination of ds.

Evidently the function $a(s)$ which we determine by the process described will be that which is expressed analytically by the integral

$$a = a_0 + \int_{s_0}^{s} \varphi(s)\, ds \tag{d}$$

Fig. 82 can be used to examplify this graphical integration. We then consider the divisions $\varphi = 5, 4, 3, \ldots$ on the lower line as given, and find by use of the divided sheet the divisions $a = 12, 16, 20, \ldots$ on the upper line.

165. Application to Two-Dimensional Scalar Fields.—The application of the described process of linear differentiation to scalar fields in two dimensions will be the most important graphical differential operation. It will return in most of the more complex differentiation-problems.

Let the two-dimensional scalar field be represented by a system of equiscalar curves

$$a = a_0 \qquad a = a_1 \qquad a = a_2 \quad \ldots \tag{a}$$

where $a_0, a_1, a_2, \ldots$ are integer values in the widened sense of the word as defined above. Let further a system of curves s be given which cut through the field. (Compare fig. 83.) The scalar a will then have a definite value at each point of a curve s. On each of these curves the scalar a will appear as a function of the length of arc s. We can therefore perform a linear differentiation along each curve s, using the divided sheet as described in the preceding section. In this manner we find the value of the derivative

$$\varphi = \frac{da}{ds} \tag{b}$$

at a great number of points. Afterwards we can draw curves for integer values of φ, and thus arrive at the common representation of the field of the scalar φ, which is the derivative of the scalar α.

In the way described we arrive at the field of φ by a discontinuous process. But it can be changed at once into a *continuous* one. Instead of moving the differentiating sheet along the curves s, we move it along the curves α = const., and measure the line-elements which are contained between two curves $\alpha = \alpha_0$ and $\alpha = \alpha_1$. When we come to places where the element ds is seen to give one of the required integer values of φ we make a mark through the slit of the sheet. In this manner we mark points through which the required curves for integer values of φ are to go. Afterwards these curves can be drawn continuously through the points determined.

Vice versa the problem of determining the field of α when that of φ is given, *i. e.*, the problem of integration, will be determinate when an initial value of α is given at

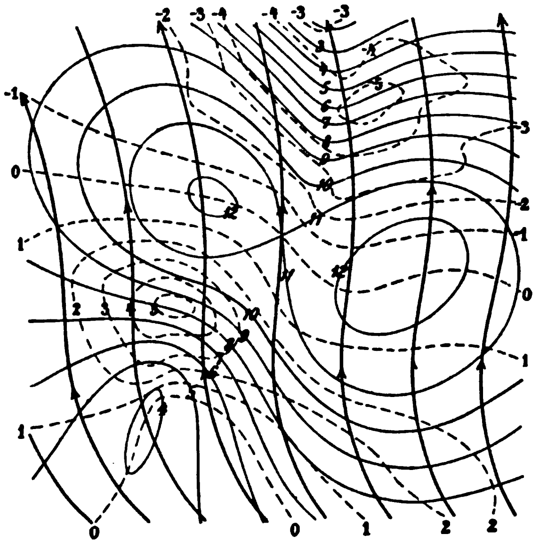


FIG. 83.—The curves s are represented by thick lines with arrow-heads; the curves $\alpha = 12, 11, 10, \ldots$ by fine continuous lines; and the curves $\varphi = \frac{d\alpha}{ds} = \ldots 2, 1, 0, -1, -2, \ldots$ by stippled lines.

one point of each curve s, for instance when an initial curve $\alpha = \alpha_0$ is given. The measurements which are to give the values of α at other points can be performed along one after another of the curves s as described in the preceding section. Or they can be performed first along the initial curve $\alpha = \alpha_0$ in order to determine the points of the next curve $\alpha = \alpha_1$; then along this second curve in order to determine the next curve $\alpha = \alpha_2$, and so on. Both methods are continuous.

It will much facilitate the drawing of the field of the derivative to observe that the curve $\varphi = 0$ can be drawn at once, without any use of the differentiating sheet; for evidently this curve will pass through all the points of tangency of the curves s with the curves α = const., including the points of maximum, minimum, or maximum-minimum, at which the curve α = const. is reduced to a point or cuts itself. Vice versa we conclude that when the field of φ is given, and that of α shall be determined by integration, the curves α = const. must have tangency with the curves s at the points where these curves are cut by the curve $\varphi = 0$.

As we shall make an extensive use of the process of differentiation described, it will be important to direct the attention to the character of the errors which will enter, and the methods of diminishing their influence. Let us for this purpose consider the derivatives of the two fields which are given by fig. 84 A and B. In both cases the lines $\alpha = \text{const.}$ have the same general course and the same *average* distance from each other; but on the first figure the distance varies in a regular way, and in the second it shows small irregularities in its variations. The curves which represent the field of the differential quotient are then seen to be very different in the two cases. In the first case they have a regular course, while in the second they show great sinuosities.

Now a free off-hand drawing which should represent a field as that of the first figure will in consequence of the unavoidable errors get more or less the character of the second figure. *Thus the irregularities in the drawing of the given field will cause oscillations in the course of the curves representing the field of the derivative.* But as the oscillations will go equally to both sides, they will be easy to reduce afterwards.

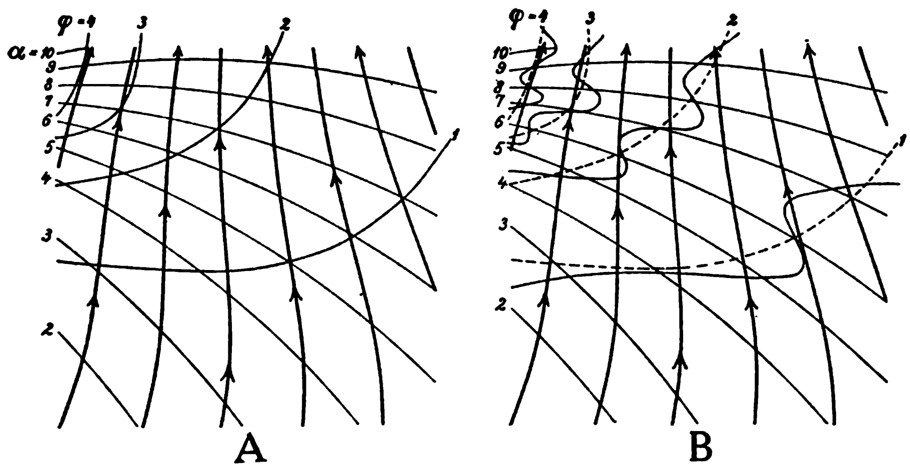


FIG. 84.—Regular course (A) and oscillating course (B) of the curves representing the differential-quotient $\varphi = \frac{d\alpha}{ds}$

A good method of diminishing them from the beginning will be to measure the line-elements not one by one, but two by two or even more of them at a time. On the divided sheet we can always find the proper hyperbolæ for doing this. But the final correction will always consist in reducing those sinuosities which are seen to arise from errors in the drawing and not from the true nature of the given field. By this correction *à posteriori* of the field of the derivative, we get a determination of this field which by far exceeds the accuracy of the single measurements upon which the process of differentiation depends.

For the process of integration, the irregularities in the drawing of the given field will cause no errors of greater importance. The process of integration itself involves a formation of averages, by which the consequences of the irregularities in the drawing are reduced.

166. Other Forms of the Problem of Linear Differentiation and Integration.—Instead of constructing an auxiliary sheet for the determination of the differen-

tial quotient itself, $\frac{da}{ds}$, of a given function a, we can construct a sheet for the determination of any function of this differential quotient

$$\varphi = f\left(\frac{da}{ds}\right) \tag{a}$$

The sheet which allows us to derive this function $\varphi(s)$ from the given function $a(s)$ will also allow us to solve the corresponding problem of integration, viz, when $\varphi(s)$ is given to determine the function $a(s)$ which is defined as function of φ by the differential equation (a).

In order to construct this auxiliary sheet we solve equation (a) with respect to $\frac{da}{ds}$ and obtain $\frac{da}{ds} = F(\varphi)$ or

$$da = F(\varphi)ds \tag{b}$$

As in the preceding case, we consider $\varphi = x$ as the abscissa and $ds = y$ as the ordinate of a point, and construct the curves $F(x)y = \ldots -2, -1, 0, 1, 2, \ldots$ to positive or negative integer values of da (fig. 85). When a value of da and a value of ds are given, we have a certain curve given on the sheet and a certain ordinate belonging to this curve. The corresponding abscissa then gives the value of the function $\varphi = f\left(\frac{da}{ds}\right)$. This gives the solution of the problem of differentiation. If on the other hand $\varphi(s)$ is given the corresponding ordinate up to a certain curve, da gives the length ds, for which we have a certain integer increase in the value of the required function a. This leads to a method of determining step by step a series of points at which the function

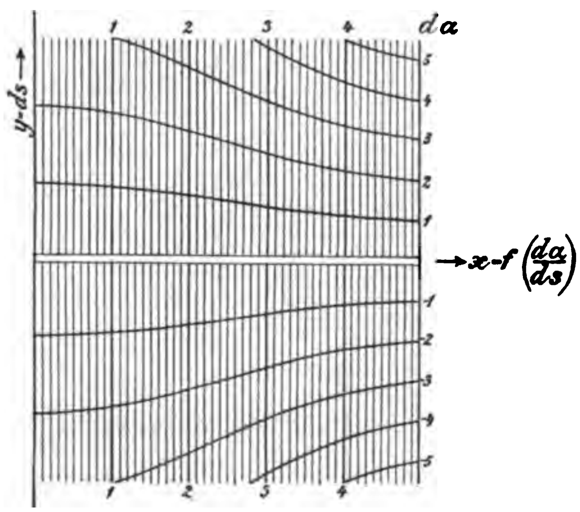


FIG. 85.—Divided sheet for determination of the field of a function $f\left(\frac{da}{ds}\right)$

$$a = a_0 + \int_{s_0}^{s} F(\varphi(s))ds \tag{c}$$

has given integer values. The procedure is precisely the same as in the preceding case.

We shall consider only one simple example. Let $f\left(\frac{da}{ds}\right) = \frac{ds}{da}$. We shall then determine

$$\varphi = \frac{ds}{da} \tag{d}$$

that is, we shall determine simply the lengths ds between the points where the function a has the integer values $1, 2, 3, \ldots$ Corresponding to equation (b) we then get

$$da = \frac{ds}{\varphi} \tag{e}$$

If we consider $\varphi = x$ as abscissa and $ds = y$ as ordinate, the constant values of da give straight lines $\frac{y}{x}$ = const. which pass through the origin of the coordinates. Fig. 86 contains these lines for "integer" values $da = -2.10^{-1}, -1.10^{-1}, 0, 1.10^{-1}, 2.10^{-1} \ldots$ We get the well-known sheet for direct measurements of lengths. When the line-elements are short, it will be convenient to measure them two by two between the lines $da = +1.10^{-1}$ and $da = -1.10^{-1}$. The abscissa will then give the length in ten-fold enlargement, with a corresponding increased accuracy of the reading. Longer elements can be measured without any enlargement by use of the lines $da = \pm 1$.

167. Curves of Equal Intensity and Curves of Equal Transport.—In order to give an example of the use of this sheet, we shall treat the following problem: to draw curves for equal transport when the vector-lines and the intensity-curves of a vector are given. The curves of equal transport will be determinate

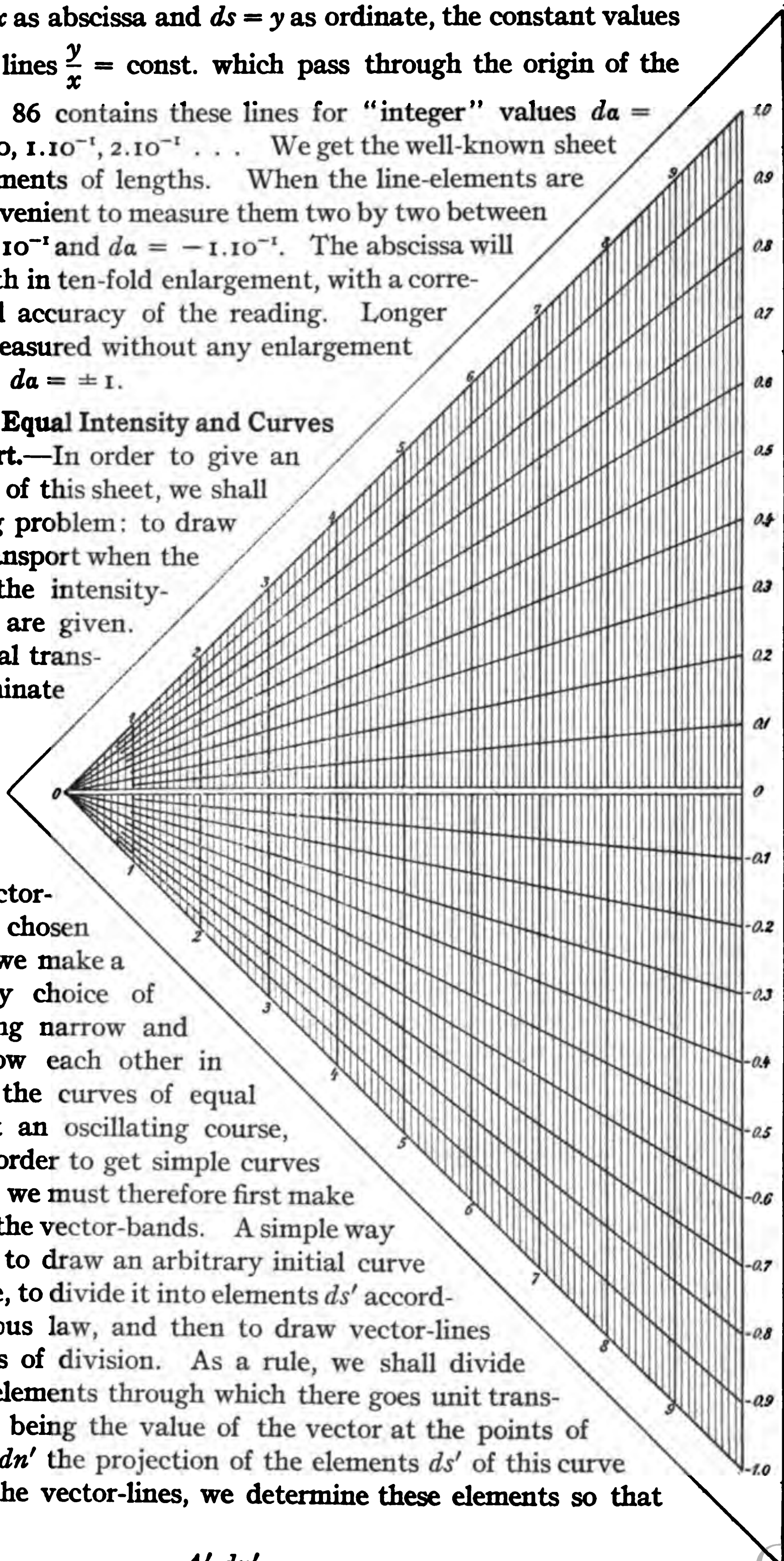


FIG. 86.—Divided sheet for direct length-measurements.

only when the vector-bands have been chosen (section 119). If we make a perfectly arbitrary choice of these bands, letting narrow and broad bands follow each other in an irregular way, the curves of equal transport will get an oscillating course, see fig. 84 B. In order to get simple curves of equal transport, we must therefore first make a careful choice of the vector-bands. A simple way of doing it will be to draw an arbitrary initial curve C' of regular shape, to divide it into elements ds' according to a continuous law, and then to draw vector-lines through the points of division. As a rule, we shall divide the curve C' into elements through which there goes unit transport; that is, A' being the value of the vector at the points of the curve C', and dn' the projection of the elements ds' of this curve on the normal to the vector-lines, we determine these elements so that for each of them

$$(a) \qquad A'\,dn' = 1$$

(see fig. 87). This principle for dividing the curve C' into elements has the advantage that it at once leads to the determination of the bands of unit transport in the cases where the vector is solenoidal.

The vector-bands being chosen, we know that the transport T is given by the product

$$(b) \qquad T = A\ dn$$

A being the intensity of the vector and dn the breadth of the band. In order to find the field of the scalar T, we have first to form the field of the line-element dn. This is done by making continuous use of the divided sheet for direct length-measurements

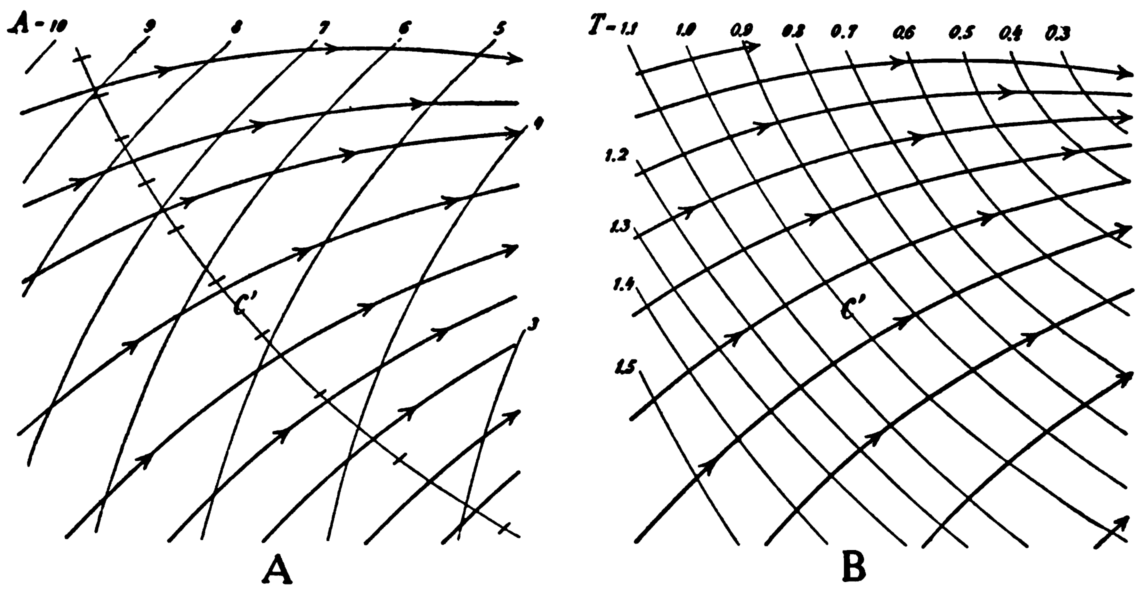


Fig. 87.

A. Vector curves s (with arrow-heads) and intensity curves $A = 10, 9, 8, 7, \ldots$ (fine continuous lines).
B. Vector curves s (with arrow-heads) and curves of equal transport $T = 1.1, 1.0, 0.9, \ldots$ (fine continuous lines).

(fig. 86). The curves n along which the line-elements dn should be measured need not be drawn; for the sheet can with the same ease be placed with its ordinates normal to as parallel to the given curves s. Afterwards the graphical multiplication of the field of dn by that of the intensity A gives the field of transport T, corresponding to the vector-bands. When the elements of the initial curve C' fulfil the condition (a) this curve will appear as the curve $T = 1$ in the field of transport.

That the use described of the divided sheet is a process of differentiation from the analytical point of view is thus seen: The choice of vector-lines by the division of the initial curve C' into elements corresponds to the choice of a continuous scalar function α which has these vector-lines for equiscalar curves $\alpha = 1, 2, 3, \ldots$ T will then be expressed by the equation:

$$(b') \qquad T = A \frac{dn}{d\alpha}$$

where $d\alpha = 1$ for the chosen interval between the successive curves.

If we wish to return from the field of transport to that of intensity of the vector A, we have to use the formula

(c) $$A = T\frac{1}{dn}$$

or corresponding to (b′)

(c′) $$A = T\frac{d\alpha}{dn}$$

We then use the common differentiating sheet for forming the field of $\frac{1}{dn}$ or $\frac{d\alpha}{dn}$ and afterwards perform the graphical multiplication of this derivative with the scalar T.

168. Differentiations of Higher Order. Curvature and Divergence of a System of Curves.—The processes described of directional or of linear differentiations can be repeated any number of times. By use of the auxiliaries which we have described

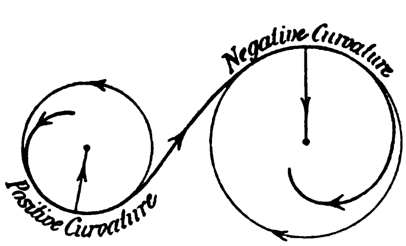


FIG. 88.—Positive and negative curvature of a curve.

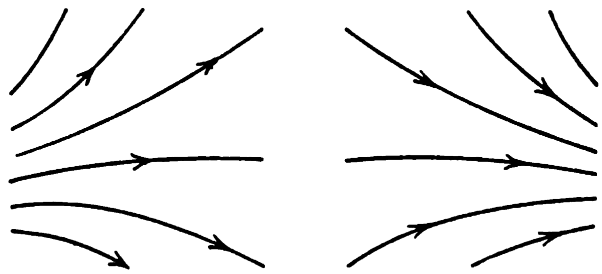


FIG. 89.—Positive and negative divergence of a system of curves.

we can thus form a derivative of any order. In precisely the same manner the process of integration can be repeated, and will then lead back to the primary function from a derivative of any order.

A case of special importance is that in which a directional differentiation is succeeded by a linear one.

In order to consider this case let us suppose that a system of curves s is given. By directional differentiation we can derive the angle α which represents the direction of the tangent to these curves and represent the field of this angle by the isogonal curves

(a) $$\alpha = \text{const.}$$

Upon the field of the scalar α we can perform a linear differentiation, which will then show the variation from place to place of the angle α. Let this linear differentiation be performed along the direction of the originally given curves s themselves. This derivative

(b) $$\gamma = \frac{d\alpha}{ds}$$

represents the change of direction of the tangent per unit length along the curve, *i. e.*, the *curvature* of the curves s. The differentiation can be performed as described in section 165 by use of the divided sheet of fig. 81, and will give the *field of curvature* of the given system of curves s.

It must be remarked that (*b*) defines curvature as a quantity which has a definite sign. This sign depends on the direction for the positive increase of the angle (sec. 155), and the positive direction along the curve *s*. It is seen at once that the rule of signs can be given in this form:

An element ds of a curve has positive or negative curvature according as it determines positive or negative circulation on the osculating circle (fig. 88).

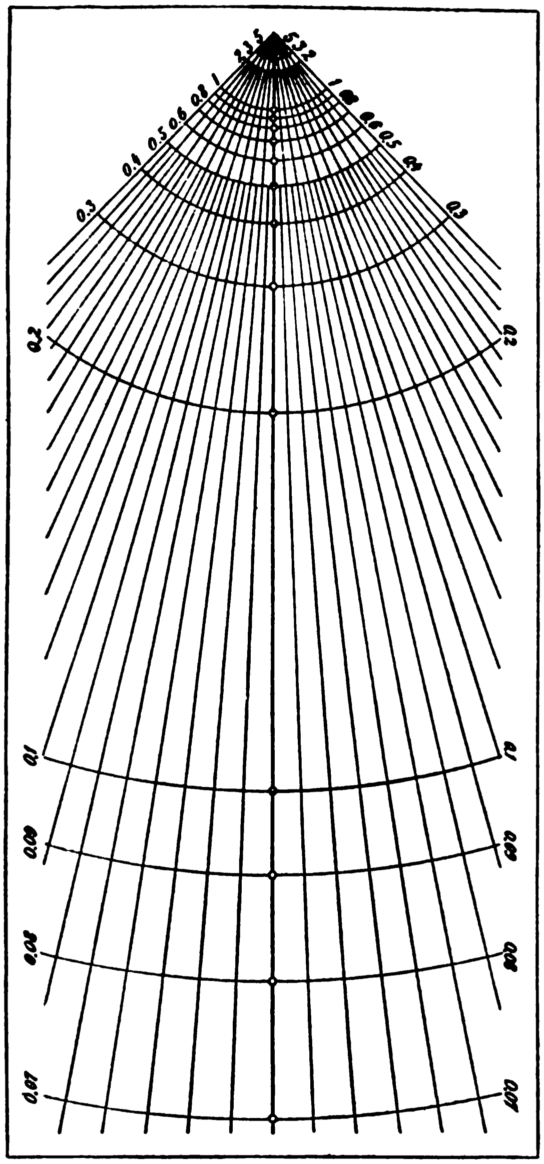


FIG. 90.—Divided sheet for the determination of curvature and divergence of curves.

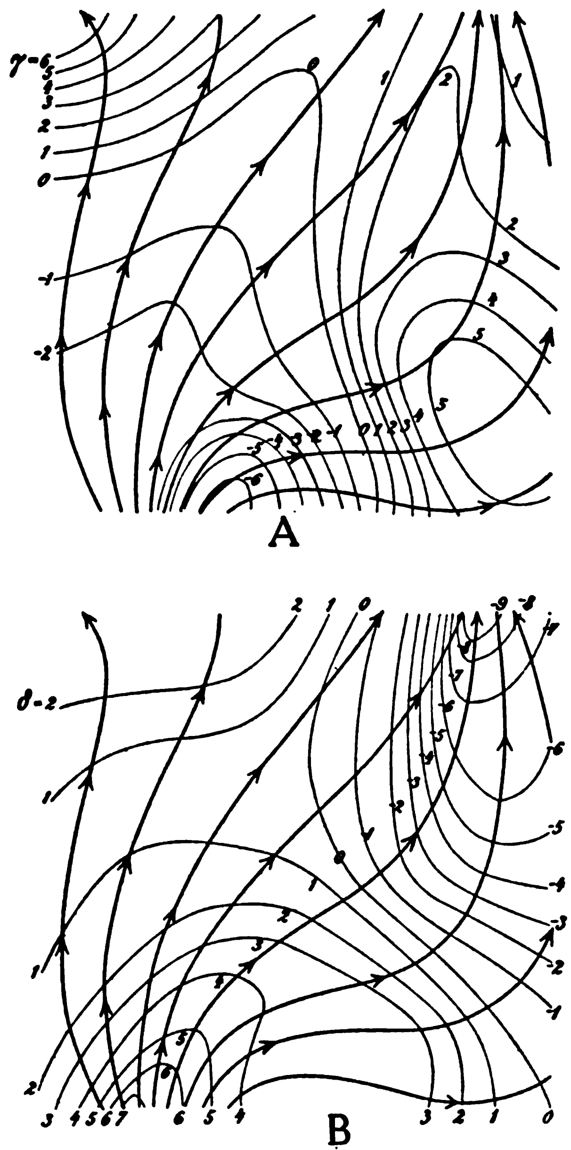


FIG. 91.—Field of curvature (A) and field of divergence (B) of a system of curves.

For our charts in horizontal projection the same rule can be stated thus:

Curvature will be positive or negative according as an observer who looks in the positive direction of the curve has the center of curvature to the left or to the right.

Or, let the linear differentiation be performed along the direction of the normal n to the curves s. This derivative

$$(c) \qquad \delta = \frac{da}{dn}$$

will represent the change of direction per unit length when we proceed *normally from curve to curve* instead of tangentially along one and the same curve s. It therefore shows how the different curves s diverge from each other. Equation (c) gives the field of divergence of the given system of curves s. This field can also be found by use of the divided sheet of fig. 81, and it will not be necessary to draw the normal curves n, as the sheet can be placed with the same ease both with its ordinates normal to and parallel to the given curves s.

When we remember our definition of the positive normal n to a given direction s (section 155) we see that formula (c) contains the following rule for the sign of the divergence δ:

> *Divergence of a system of curves will be positive or negative according as they appear to an observer looking in the positive direction of the curves to diverge or to converge* (fig. 89).

As will be seen at once, there is a close relationship between the fields of divergence and of curvature. The field of divergence of a system of curves is the field of curvature to the normal curves, and vice versa the field of curvature is the field of divergence to the normal curves.

The derivatives (b) and (c) are derivatives of the second order in reference to the originally given system of curves. The two successive operations, consisting in a directional and a subsequent linear differentiation, can be combined into one which represents a differentiation of the second order and which can be performed by the divided sheet of fig. 90. This sheet contains a set of concentric circles with integer values (multiplied by a power of 10) of the curvature, *i.e.*, integer reciprocal values of the length of the radii and a set of divergent radii with equal and small angular intervals. For continuous use the sheet is perforated at the points of intersection of the circles with the central radius.

This sheet can be placed directly upon the field of the system of curves s originally given. In order to find the field of curvature (fig. 91 A) we place it with the circles tangential to and the radii normal to the curves s. One after another of the curves s is followed, and the points are marked where these curves give complete osculation with one of the circles of the sheet. In order to find the field of divergence (fig. 91 B) we place the sheet with the circles normal to and the radii tangential to the curves s. One after another of the curves s is followed, and the points are marked where the circles osculate the normal curves, *i. e.*, the points where one of the circles is normal to the curves next to the considered curve s. As a supplementary condition we have that the radii shall be tangential to the curves at the points where these radii are cut by the circles.

169. Partial Derivatives; Ascendant and Gradient.—The two-dimensional scalar a is a function of two coordinates which figure as independent variables.

Now let us consider the curves s as the one set of coordinate-curves. The derivative of a scalar a with respect to s will then be the one partial derivative of the dependent variable a. It is a special case that the curves s are parallel and equidistant straight lines. If we use two such systems of lines which are normal to each other, and call the length of arc along the one set x, and along the other set y, the two partial derivatives will be

$$(a) \qquad F_x = \frac{\partial a}{\partial x} \qquad F_y = \frac{\partial a}{\partial y}$$

The fields of these partial derivatives can be determined by use of the divided sheet of fig. 81.

The two partial derivatives are the rectangular components of the *ascendant* **F** of the scalar. As we have shown already (Statics, section 17), this vector is directed along the normal n to the equiscalar curves a = const., and is numerically equal to the derivative of a with respect to the length of arc n measured along the normal curves

$$(b) \qquad F = \frac{da}{dn}$$

In order to abbreviate we shall introduce here a useful notation. The fact that the vector **F** is in the defined relation to the scalar a will be expressed by the single vector-equation

$$(c) \qquad \mathbf{F} = \nabla a$$

This equation is by definition equivalent to the two scalar equations (a), and in the case of the three-dimensional field it will be equivalent to three such equations. A vector **G** of the opposite direction

$$(d) \qquad \mathbf{G} = -\nabla a$$

represents the gradient of the scalar a.

The field of the ascendant or of the gradient can be found by algebraic methods (section 157) from the fields of the two rectangular components; but it can also be derived directly from the field of the given scalar a. This direct method will involve separate determinations of the direction and of the magnitude of the vector.

The vector-lines can be drawn at once as normal curves to the equiscalar curves a = const. If we wish to have the direction represented by isogons, we have to use the directional differentiation described in section 163, and to give the isogons such numbers that they represent the direction of the normal curves n, not of the equiscalar curves a = const.

The intensity-field of ascendant or gradient are found by use of the differentiating sheet of fig. 81 in accordance with formula (b). The field will contain no zero-curve. It will only have zero-points at the points of maximum, minimum, and maximum-minimum of the scalar a. These zero-points will be singular points of intersection of the vector-lines as well as of the isogons. As points for absolute minimum of the scalar value of the vector they will be surrounded by closed curves of equal intensity. The drawing of the field is therefore very much facilitated by the circumstance that these zero points are given beforehand.

Fig. 92 represents the ascendant of the same field of which fig. 83 represents a partial derivative.

From the field of the ascendant (a) we can derive that of any other derivative

(e) $$F_s = \frac{\partial \alpha}{\partial s}$$

as we have

(f) $$F_s = F \cos \theta$$

where θ is the angle between the directions n and s. This algebraic method of finding the partial derivative F_s will be convenient if the direction of the ascendant **F** is represented by isogonal curves φ = const., and the direction of s by isogonal curves σ = const. We shall then pass from the field of **F** to that of F_s by the following operations (compare sections 149, 156).

(1) By graphical subtraction we form the field of the angle $\theta = \varphi - \sigma$. In the drawing of these auxiliary curves special attention should be attached to the drawing of the curves $\theta = 16$, and $\theta = 48$, which will be curves $F_s = 0$ in the resultant field.

(2) By use of these auxiliary curves and the curves F = const., we derive the field of the scalar value of F_s according to equation (c) by use of the first of tables K.

By this process we can thus derive the partial derivative of fig. 83 from the ascendant-field of fig. 92.

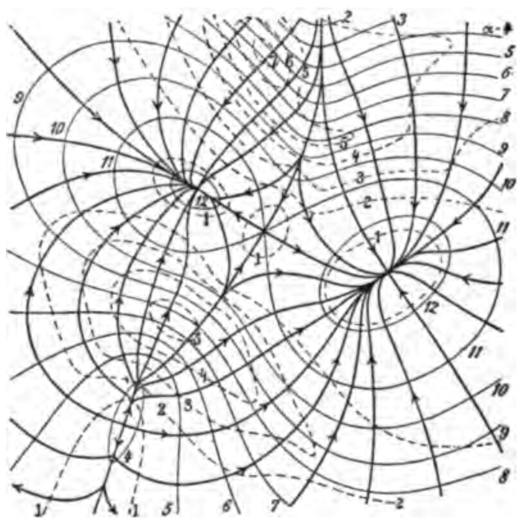


FIG. 92.—Scalar field $\alpha = 12, 11, 10, \ldots$ (fine continuous lines), vector-lines of the ascendant (thick lines with arrow-heads), and intensity-curves of the ascendant (stippled curves).

If we know the field of the ascendant or the gradient and the value of the scalar α at a single point, we can reconstruct the field of the scalar α. The simplest method will be this: We first perform a linear integration along the particular curve n which passes through the point where we have the known value of α. By this integration we find a series of points through which equiscalar curves representing the required integer values of α shall pass. Through these points we may then by directional integration draw the curves α = const., normal to the vector-curves of the ascendant or the gradient.

170. Divergence of a Two-Dimensional Vector.—We have considered already the "transport" in the two-dimensional field (section 119), *i. e.*, the integral of the normal component A_n of a vector taken along a curve.

(a) $$\int A_n \, ds$$

In the special case of a closed curve the transport directed outward was called the "outflow" out from the area limited by the curve.

This outflow has a simple additive property. Let the considered area be divided by a line into two parts (fig. 93). The transport through the dividing line will then appear in the expression for the outflow out of each part. But in the sum of these two outflows this transport will drop out, as it represents the transport *out of* the one and *into* the other area. The sum of the outflows out of the two parts will therefore be equal to the outflow out of the total area. As each part can be divided again, and so on, we get the general result that the outflow out of all the parts into which an area can be divided will be equal to the outflow out of the total area. We symbolize this result by the equation

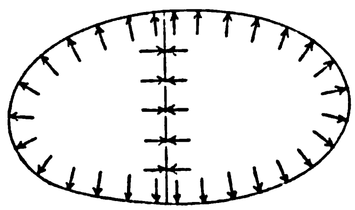


Fig. 93.

(*b*) $$\int A_n\, ds = \Sigma \int A_n\, ds$$

the first member being extended to the contour of the total area, and the integrals in the second member being extended to the contours of all the parts into which the total area has been divided.

The division may be continued indefinitely. The areas of which the contours appear in the second member of equation (*b*) may therefore be considered as elementary areas $d\sigma$. As they can have any form let them be limited by the two elements ds and ds' of two vector-lines, and by the two elements dn and dn' which are normal to these lines (fig. 94). The outflow will be the difference between the transport $A'dn'$ and Adn through the latter elements.

(*c*) $$A'dn' - Adn$$

Here A' will vary as we proceed along a vector-line s, and the same will be the case with the normal distance dn' between the two vector-lines. We may then consider these quantities as functions of s and use the developments

$$A' = A + \frac{\partial A}{\partial s} ds \qquad dn' = dn + \frac{\partial dn}{\partial s} ds$$

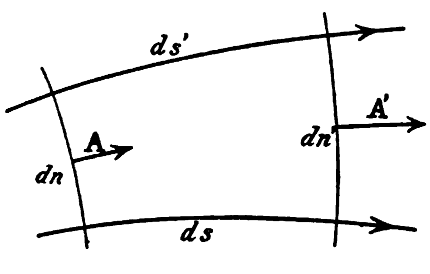


Fig. 94.

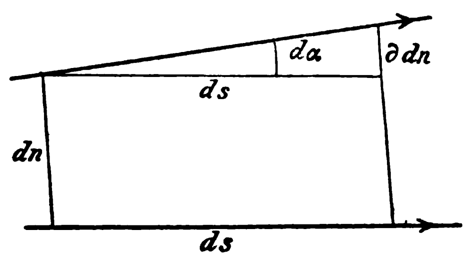


Fig. 95.

When we introduce this and disregard quantities of the second order we get as expression of the outflow

(*c'*) $$\frac{\partial A}{\partial s} ds dn + A \frac{\partial dn}{\partial s} ds$$

or, when we separate the factor $d\sigma = dnds$ which represents the area of the element, we get the expression of the outflow in the form

(*c''*) $$\left(\frac{\partial A}{\partial s} + A \frac{1}{dn} \frac{\partial dn}{\partial s}\right) d\sigma$$

We have thus expressed the outflow through the contour of an elementary area as the product of the area of the elements and a factor which must then represent the *outflow per unit area*. We shall call this outflow per unit area the *two-dimensional divergence* of the vector **A** and introduce the notation

(*d*)
$$\text{div}_2 \mathbf{A} = \frac{\partial A}{\partial s} + A \cdot \frac{1}{dn} \frac{\partial dn}{\partial s}$$

We can now write every term in the sum which forms the second member of equation (*b*) in the form $\text{div}_2 \mathbf{A}\, d\sigma$. The sum then takes the form of an integral extended to all the elements of area $d\sigma$; that is, we get the formula

(*e*)
$$\int A_n ds = \int \text{div}_2 \mathbf{A}\, d\sigma$$

or expressed in words:

> *The integral of the normal component of a two-dimensional vector taken around a closed curve is equal to the integral of the two-dimensional divergence of this vector taken over the area bordered by the closed curve.*

The two-dimensional divergence, or the outflow per unit area, can be found by a process of differentiation given by equation (*d*). The last term has a simple geometrical sense. As dn represents the elementary distance between two curves s, the derivative $\frac{\partial dn}{\partial s}$ will evidently represent the elementary angle $d\alpha$ between the tangents of two curves s which have the distance dn from each other (see fig. 95). Thus we get

(*f*)
$$\frac{1}{dn} \frac{\partial dn}{\partial s} = \frac{d\alpha}{dn} = \delta$$

where δ is the divergence of the system of curves s as defined in section 168. Thus the two-dimensional divergence of the vector **A** can be written in the form

(*g*)
$$\text{div}_2 \mathbf{A} = \frac{\partial A}{\partial s} + A\delta$$

where δ is the divergence of the vector-lines. When in this formula we give the vector **A** the constant scalar value 1, we get $\text{div}_2 \mathbf{A} = \delta$, which shows that the divergence of a system of curves is equal to the divergence of a unit vector which has these curves as vector-lines.

By formula (*g*) we have reduced the construction of the field of divergence of a two-dimensional vector to graphical differentiations which we have performed already. We shall find it by the following series of operations:

(1) We perform the graphical differentiation of the intensity-field of the given vector with respect to its vector-lines. (See fig. 83, where we can interpret the curves s as the vector-lines and the given scalar field as the intensity-field of the given vector.)

(2) We form the field of divergence of the vector-lines, using either of the two developed methods according as the isogons of the vector or the vector-lines themselves are given. (See section 168.)

(3) We perform the graphical multiplication of the intensity-field of the vector and the divergence-field of its vector-lines.

(4) We perform the graphical addition of the two fields obtained by the operations (1) and (3).

The construction described will be of great importance for the kinematic diagnosis of air- and sea-motions.

Other expressions of the divergence will also be useful. If the vector-lines happen to run at an invariable distance dn from each other, we shall have the divergence of the vector-lines equal to zero, and the divergence of the vector will be given by one term only, $\frac{\partial A}{\partial s}$. Now, when we express the field of the vector **A** by the fields of its two cartesian components A_x and A_y, the component-fields have straight and parallel lines of flow. The divergence of the two component-fields will be respectively $\frac{\partial A_x}{\partial x}$ and $\frac{\partial A_y}{\partial y}$ and their sum will give the divergence of the resultant field.

(h) $$\mathrm{div}_2\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y}$$

When the fields of the rectangular components A_x and A_y are given, this expression gives a simple construction of the fields of divergence. By graphical differentiation we form separately the fields of $\frac{\partial A_x}{\partial x}$ and of $\frac{\partial A_y}{\partial y}$, and then by graphical addition that of $\mathrm{div}_2\mathbf{A}$.

Inasmuch as coordinate-methods should be used on our charts, it must be remembered that the meridians are not equidistant coordinate-curves. The divergence of the meridians must be taken into account when the divergence of a velocity-field should be formed separately from charts of the south–north component and of the west–east component of the wind.

171. Divergence of a Vector in Space.—The two-dimensional divergence which we can represent on our charts will have its importance as part of the three-dimensional divergence of that vector in space of which the two-dimensional vector is a component. We shall therefore also consider the divergence of a vector in space.

Transport in the vector-field in space is represented by the surface-integral of the normal component of the vector

(a) $$\int A_n d\sigma$$

In the case when the surface σ is closed the transport will represent the outflow of the volume bounded by the closed surface (see section 111).

If we divide a given volume into any number of parts and form the sum of the outflows out of each part, the transport through the dividing surfaces will cancel, and we find that the outflow in three dimensions has the same additive property as it has in two dimensions. This property can be expressed by the formula

(b) $$\int A_n d\sigma = \Sigma \int A_n d\sigma$$

where the integral appearing as the first member is extended to the limiting surface of the total volume, and the integrals appearing in the second member are extended to the limiting surfaces of the different parts into which the total volume is divided.

Now let the total volume be divided into elementary volumes, consisting of infinitely short trunks of infinitely narrow vector-tubes. There will be a transport only through the surface-elements $d\sigma$ and $d\sigma'$ which form sections of the tube (fig. 96). These sections being normal, we get the transport through them equal respectively to $Ad\sigma$ and $A'd\sigma'$, and the outflow equal to their difference

$$A'd\sigma' - Ad\sigma \tag{c}$$

Here we can develop A' and $d\sigma'$ as functions of the length of arc s along the axis of the tube

$$A' = A + \frac{\partial A}{\partial s}ds \qquad d\sigma' = d\sigma + \frac{\partial d\sigma}{\partial s}ds$$

When we introduce this and leave the term of the second order out of consideration, we get the expression of the elementary outflow (c) in the form

$$\frac{\partial A}{\partial s}\, ds d\sigma + A\,\frac{\partial d\sigma}{\partial s}\, ds \tag{c'}$$

Introducing the volume of the element $d\tau = d\sigma ds$, this expression may be written

$$\left(\frac{\partial A}{\partial s} + A\cdot\frac{1}{d\sigma}\,\frac{\partial d\sigma}{\partial s}\right)d\tau \tag{c''}$$

Thus for elementary volumes the outflow is proportional to the volume of the element. The factor of proportionality represents the outflow per unit volume, and

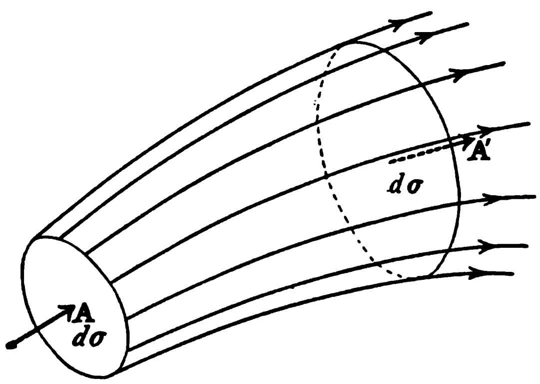


FIG. 96.

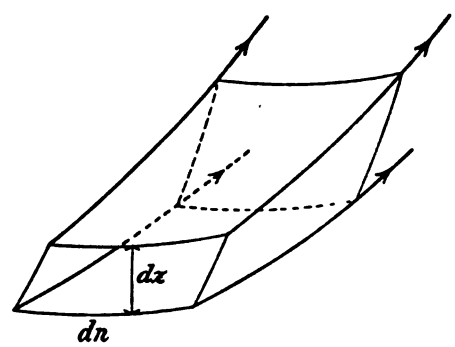


FIG. 97.

is called the *three-dimensional divergence* or simply the *divergence* of the vector **A**

$$\operatorname{div}\mathbf{A} = \frac{\partial A}{\partial s} + A\cdot\frac{1}{d\sigma}\,\frac{\partial d\sigma}{\partial s} \tag{d}$$

As we can now write each term in the second member of equation (b) in the form div **A** $d\tau$, this second member takes the form of a sum extended to all the elements of volume $d\tau$, *i. e.*, the form of a volume-integral. We thus get the important formula

$$\int A_n d\sigma = \int \operatorname{div}\mathbf{A}\, d\tau \tag{e}$$

or in words:

> *The integral of the normal component of a vector taken over a closed surface is equal to the volume-integral of the divergence of the vector taken in the volume limited by the closed surface* (Gauss's theorem).

This theorem allows us to bring the solenoidal condition—section 112 (*a*)—into a new form; for when the surface-integral in equation (*e*) is zero for every closed surface in the field, the volume-integral must also be identically zero, and this involves

$$\text{div } \mathbf{A} = 0 \tag{f}$$

This is the differential form of the solenoidal condition.

The expression $\frac{1}{d\sigma}\frac{\partial d\sigma}{\partial s}$ which appears in the equation of definition (*d*) has a similar significance in space as $\frac{1}{dn}\frac{\partial dn}{\partial s}$ in two dimensions. When the area $d\sigma$ of the cross-section of the tube is constant, the considered trunk of the tube may be compared to a cylinder. When $d\sigma$ varies, the trunk of the tube may be compared to a cone, and the derivative $\frac{\partial d\sigma}{\partial s}$ will represent its solid angle. Then $\frac{1}{d\sigma}\frac{\partial d\sigma}{\partial s}$ will represent the ratio of this solid angle to the cross-section of the tube and thus be a measure of what we may call the divergence of the curves *s* in space.

In order to express this divergence by the corresponding divergences in two dimensions we will consider vector-tubes which are produced in the usual way by the intersection of two sets of surfaces of flow (fig. 97). Each tube will then have the well-known parallelogrammatic cross-section. If dn is one side in the parallelogram, and dz the corresponding height, we have $d\sigma = dndz$, and get

$$\frac{1}{d\sigma}\frac{\partial d\sigma}{\partial s} = \frac{1}{dn}\frac{\partial dn}{\partial s} + \frac{1}{dz}\frac{\partial dz}{\partial s}$$

Introducing this in equation (*d*), we get this more developed form of the divergence

$$\text{div } \mathbf{A} = \frac{\partial A}{\partial s} + A \cdot \frac{1}{dn}\frac{\partial dn}{\partial s} + \mathrm{A} \cdot \frac{1}{dz}\frac{\partial dz}{\partial s} \tag{g}$$

The divergence is here given by a trinomial expression, the first two terms of which are seen to express the two-dimensional divergence—equation (*d*) of the preceding section—of the vector **A** in the surface which contains the curves *s* and *n*.

If we resolve the given vector-field into three component-fields, each with vector-lines coinciding with one set of coordinate-curves of a system of curvilinear orthogonal coordinates, we can write the divergence of each component-field in either of the forms (*d*) or (*g*). In the special case of a cartesian system the vector-lines of each component-field are straight and parallel. Each vector-tube will have a constant cross-section $d\sigma$, or constant base dn and height dz, and only the first term in the second member of formulæ (*d*) or (*g*) will be different from zero. Therefore, if we call the vectors of the three component-fields $\mathbf{A}_x$, $\mathbf{A}_y$, $\mathbf{A}_z$, and the lengths of arc measured along the vector-lines *x*, *y*, and *z*, we get for the divergence in each component-field

$$\text{div } \mathbf{A}_x = \frac{\partial A_x}{\partial x} \qquad \text{div } \mathbf{A}_y = \frac{\partial A_y}{\partial y} \qquad \text{div } \mathbf{A}_z = \frac{\partial A_z}{\partial z}$$

When we form the sum, we get the divergence of the resultant-field

$$\text{div } \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \tag{h}$$

This is the most generally used expression of the divergence of a vector in space.

When we compare with the formula (h) of the preceding section, we see that we can write the equation

$$(j) \qquad \operatorname{div} \mathbf{A} = \frac{\partial A_z}{\partial z} + \operatorname{div}_2 \mathbf{A}$$

where $\operatorname{div}_2 \mathbf{A}$ represents the divergence of that two-dimensional vector which has the components A_x and A_y. Now let the three-dimensional vector **A** be solenoidal, $\operatorname{div} \mathbf{A} = 0$. Then equation ($i$) gives

$$(j) \qquad \frac{\partial A_z}{\partial z} = -\operatorname{div}_2 \mathbf{A}$$

This is a differential equation by which we may determine the third component A_z of a solenoidal vector, of which we know the two components A_x and A_y. This will be our most important diagnostic formula. We shall use it to derive the vertical motion from the observed horizontal motion in the atmosphere.

172. Curl of a Two-Dimensional Vector.—Instead of the integral of the normal component A_n we shall now consider that of the tangential component A_s taken along a curve s.

$$(a) \qquad \int A_s ds$$

In the special case of a closed curve we shall call this integral the *circulation* of the vector **A** around the curve s. Lord Kelvin has introduced this name for cases where the vector **A** represents velocity. We shall use it, precisely as the expressions transport and outflow, even for cases of abstract vectors, which have nothing to do with motion. Circulation is a quantity which has a definite sign depending upon the direction which we have chosen as positive for rotating motion around a point or circulating motion around a closed curve (section 155).

Circulations have an additive property similar to outflows. We can join two points of the circuit originally given by a curve. The area limited by the first circuit will then be divided into two areas. We can form the sum of the circulations around the contours of each of them, using in both cases the same direction of circulation. In this sum the line-integral taken along the dividing curve will appear twice with opposite signs in the two cases, and will therefore drop out (fig. 98). Thus the sum of the circulations around the contours of the two parts of an area will be equal to the circulation around the contour of this total area. As we can continue the subdivision, we arrive at the result that the circulation around the contour of any area is equal to the sum of the circulations around the contours of all the areas into which it can be subdivided. We can express this result by the equation

$$(b) \qquad \int A_s ds = \Sigma \int A_s ds$$

extending the integral of the first member to the contour of the primary area and the integrals of the second to the contours of the areas produced by the division.

Now let the primary area be subdivided into elementary areas by two systems of curves, namely, the vector-lines and their positive normal curves n. The elements dn of the contour of these areas will then give no addition to the line-integral. The circulation in positive direction around the contour (fig. 99) will be represented by the difference

$$(c) \qquad -(A'ds' - Ads)$$

A' and ds' will vary as we proceed along the curve n. They can therefore be developed as functions of the length of arc n

$$A' = A + \frac{\partial A}{\partial n} dn \qquad ds' = ds + \frac{\partial ds}{\partial n} dn$$

Introducing this and leaving the term of second order out of consideration, we get for (c)

$$-\left(\frac{\partial A}{\partial n} dnds + A \frac{\partial ds}{\partial n} dn\right)$$

or introducing the area $d\sigma = dnds$ of the element

$$(c') \qquad -\left(\frac{\partial A}{\partial n} + A \cdot \frac{1}{ds} \frac{\partial ds}{\partial n}\right) d\sigma$$

The factor of $d\sigma$ then represents the circulation per unit area, or the *curl* of the two-dimensional vector **A**. We shall introduce the notation

$$(d) \qquad \text{curl}_2 \mathbf{A} = -\left(\frac{\partial A}{\partial n} + A \frac{1}{ds} \frac{\partial ds}{\partial n}\right)$$

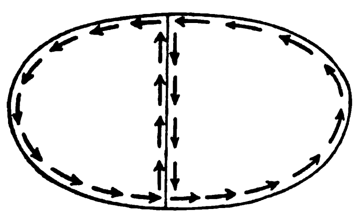


Fig. 98.

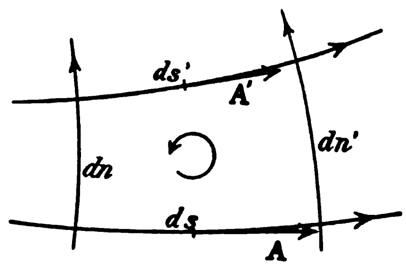


Fig. 99.

the suffix 2 denoting that the operation curl is performed only in two dimensions. We shall see presently that the curl of the three-dimensional vector is a vector. But, precisely as in the case of the vector-product, the vector-nature of the curl does not appear if we confine ourselves to the consideration of two-dimensional fields.

We can now write every term in the sum appearing as second member of equation (b) in the form $\text{curl}_2 \mathbf{A}\, d\sigma$. This sum then takes the form of an integral extended to the area formed by all the elements $d\sigma$. Thus we get the formula

$$(e) \qquad \int A_s ds = \int \text{curl}_2 \mathbf{A} d\sigma$$

that is, the line-integral of the tangential component of a two-dimensional vector taken around a closed curve is equal to the integral of the curl of the vector taken over the area bounded by the closed curve.

As the expression $\frac{1}{dn}\frac{\partial dn}{\partial s}$ represented the divergence of the vector-lines, section 170 (f), *i. e.*, the curvature of their positive normal curves, the expression $\frac{1}{ds}\frac{\partial ds}{\partial n}$ will represent the divergence of the positive normal curves, *i. e.*, *the negative curvature* $(-\gamma)$ *of the vector-lines* which are the negative normal curves to the curves n (section 168). That is, we can write the expression of $\text{curl}_2 \mathbf{A}$

$$(f) \qquad \text{curl}_2 \mathbf{A} = -\frac{\partial A}{\partial n} + A\gamma$$

By the expression (*f*) we can construct the field of $\text{curl}_2\,\mathbf{A}$. The construction will be perfectly analogous to that of the divergence:

(1) We perform the graphical differentiation of the intensity-field of the given vector with respect to the positive normal curves to the vector-lines.

(2) We form the field of curvature of the vector-lines of the given vector (see section 168).

(3) We perform the graphical multiplication of the latter field with the intensity-field of the given vector.

(4) We perform the graphical subtraction of the two fields obtained by the operations (3) and (1).

The expression (*f*) may be used also for forming the curl of any component of the given vector. If we use cartesian coordinates, the vector-lines of each component-field will be straight lines. The curvature γ will be equal to zero and the curl of each component-field will be expressed by the first term only. Observing the rule of signs, we get $-\frac{\partial A_x}{\partial y}$ for the curl of the component A_x, and $\frac{\partial A_y}{\partial x}$ for the field of the component A_y. Forming the sum we get

$$(g) \qquad \text{curl}_2\,\mathbf{A} = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}$$

When the field of each component is given, we can construct the field of the curl in accordance with this formula. By linear differentiation of the field of A_y along lines parallel to the axis of X, and of the field of A_x along lines parallel to the axis of Y we form the fields of the two derivatives $\frac{\partial A_y}{\partial x}$ and $\frac{\partial A_x}{\partial y}$. Afterwards, by graphical subtraction of the latter from the former, we get the field of the curl.

173. Curl of a Vector in Space.—Now let $\mathbf{A}$ be any vector in space. We may then define a vector $\mathbf{c}$ which has the rectangular components

$$(a) \qquad c_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \qquad c_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \qquad c_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}$$

By this definition we see that $\mathbf{c}$ is a vector of which each component is the curl of a two-dimensional vector: c_x of that which has the components A_z and A_y; c_y of that which has the components A_x and A_z; c_z of that which has the components A_y and A_x. We see further that each component of the vector $\mathbf{c}$ is normal to that plane which contains the two-dimensional vector from which it is derived. We will agree to represent this vector by curl $\mathbf{A}$, thus

$$(a') \qquad \mathbf{c} = \text{curl}\,\mathbf{A}$$

Now let us consider any surface σ in the three-dimensional field. The vector $\mathbf{A}$ will determine a two-dimensional vector in this surface, for which we can write the theorem (*e*) of the preceding section. But what we have written there as $\text{curl}_2\,\mathbf{A}$, conceiving $\mathbf{A}$ as the two-dimensional vector contained in the surface, may now be

expressed as the normal component to the surface of the vector (*a*), (curl **A**)$_n$. Thus

$$\int A_s ds = \int (\text{curl } \mathbf{A})_n d\sigma \tag{b}$$

or in words:

> *The line-integral of the tangential component of any vector taken around a closed curve is equal to the surface-integral of the normal component of the curl of the vector taken over any surface which has the given closed curve as contour.* (Stokes's theorem.)

As long as we deal with two-dimensional vectors only, the vector-nature of the curl does not become apparent, as we have then to deal only with the component of the vector normal to the surface which contains the two-dimensional vector-field. In this respect the case is analogous to that of the vector-product.

The general theorem allows us to demonstrate an important property of every vector which is the curl of another vector. If the surface σ is closed, the contour s will disappear, and thus the line-integral around this be zero. We then get the equation

$$\int (\text{curl } \mathbf{A})_n d\sigma = 0 \tag{c}$$

where the integral is extended to the closed surface. But this equation indicates that the vector curl **A** is a solenoidal vector. This result can also be verified if we substitute the expressions of the components (*a*) of the curl into the solenoidal condition in its differential form. This leads to the identity

$$\text{div curl } \mathbf{A} = 0 \tag{c'}$$

Thus: *The curl of a vector is a solenoidal vector.*

174. Complex Differential Operations.—Divergence and curl may be considered as the intrinsic derivatives of a vector-field. The intrinsic structure of a field is known when we know curl and divergence.

Besides the differential operations leading to these intrinsic derivatives, we shall have to consider also a differential operation of a more complex nature. **A** and **B** being two vectors, we shall consider a vector **F** which has the three components

$$\begin{aligned} F_x &= B_x\frac{\partial A_x}{\partial x}+B_y\frac{\partial A_x}{\partial y}+B_z\frac{\partial A_x}{\partial z} \\ F_y &= B_x\frac{\partial A_y}{\partial x}+B_y\frac{\partial A_y}{\partial y}+B_z\frac{\partial A_y}{\partial z} \\ F_z &= B_x\frac{\partial A_z}{\partial x}+B_y\frac{\partial A_z}{\partial y}+B_z\frac{\partial A_z}{\partial z} \end{aligned} \tag{a}$$

Remembering the definitions of the scalar product and of the ascendant, we see that the expression of each component may be written as the scalar product of the vector **B** and the three ascendants ∇A_x, ∇A_y and ∇A_z, thus

$$F_x = \mathbf{B}.\nabla A_x \qquad F_y = \mathbf{B}.\nabla A_y \qquad F_z = \mathbf{B}.\nabla A_z$$

We will denote the vector which has the components (*a*) by the sign $\mathbf{B}\nabla\mathbf{A}$, thus

$$(a') \qquad \mathbf{F} = \mathbf{B}\nabla\mathbf{A}$$

The vector-equation (*a'*) may be considered as a shortened symbolic expression of the three scalar equations (*a*).

We shall consider especially the two-dimensional vector **F** in the case when $\mathbf{B} = \mathbf{A}$. This vector will have the two components

$$(b) \qquad F_x = A_x\frac{\partial A_x}{\partial x}+A_y\frac{\partial A_x}{\partial y} \qquad F_y = A_x\frac{\partial A_y}{\partial x}+A_y\frac{\partial A_y}{\partial y}$$

and will in accordance with (*a'*) be represented by the vector-formula

$$(b') \qquad \mathbf{F} = \mathbf{A}\nabla\mathbf{A}$$

If the fields of the two components A_x and A_y are given separately, we can form the fields of F_x and F_y in accordance with these formulæ, performing for each of them two graphical differentiations, two graphical multiplications, and one graphical addition.

In order to examine more closely the relation of the derived vector **F** to the given vector **A**, we can make a special choice of the system of coordinates (fig. 100). At the considered point the axis of X shall be tangential to the vector-line s of the given vector **A**. F_x will then be the same as the component F_s tangential to the line s. As at the considered point $A_x = A$ and $A_y = 0$, and as ultimately dx will be identical with ds, we get for the tangential component

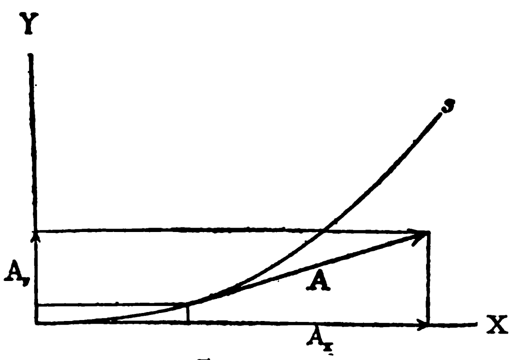


Fig. 100.

$$(c) \qquad F_s = A\frac{\partial A}{\partial s} = \frac{\partial}{\partial s}\left(\tfrac{1}{2}A^2\right)$$

As the curve s near the point of tangency forms the infinitely small angle α with the axis of x, we can write here $A_y = A\alpha$. Derivation then gives $\frac{\partial A_y}{\partial x} = \alpha\frac{\partial A}{\partial x} + A\frac{\partial \alpha}{\partial x}$

As at the point of tangency α is zero, we get here $\frac{\partial A_y}{\partial x} = A\frac{\partial \alpha}{\partial x} = A\frac{\partial \alpha}{\partial s}$. But $\frac{\partial \alpha}{\partial s}$ represents the curvature γ of the curve s. Instead of $\frac{\partial A_y}{\partial x}$ in the second equation (*b*) we can thus write $A\gamma$. When we introduce this, and remember that at the considered point $A_x = A$ and $A_y = 0$, we get this expression of F_y or F_n

$$(c') \qquad F_n = A^2\gamma$$

Thus the derived vector **F** will have two rectangular components, one which has the direction of the given vector and is equal to the derivative of the half square of the intensity of this vector with respect to its vector-lines, while the other is normal to the given vector and equal to the square of the intensity of this given vector multiplied by the curvature of its vector-lines. Hence we can form the field of this derived vector **F** by the following construction:

(1) We form the half square of the intensity-field of the given vector (section 147) and then the derivative (c) with respect to the vector-lines.

(2) We form the field of curvature of the given vector-lines (section 168) and perform the graphical multiplication of this field by that of the square of the intensity (c').

(3) We perform the graphical addition of two mutually normal vectors (section 157): the vector $\mathbf{F}_s$ which has the same direction as the given vector $\mathbf{A}$ and the intensity determined by the operation (1); and of the vector $\mathbf{F}_n$ which is normal to the given vector and has the intensity determined by the operation (2).

We can also give another method for determining the vector $\mathbf{F}$. We can change the second member of equations (b): in the first of these equations by adding and subtracting the term $A_y\frac{\partial A_y}{\partial x}$; in the second by adding and subtracting $A_x\frac{\partial A_x}{\partial y}$. This gives

$$(d)\quad \begin{aligned} F_x &= A_x\frac{\partial A_x}{\partial x}+A_y\frac{\partial A_y}{\partial x}-A_y\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)=\frac{\partial}{\partial x}\left(\tfrac{1}{2}(A_x^2+A_y^2)\right)-A_y\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)\\ F_y &= A_x\frac{\partial A_x}{\partial y}+A_y\frac{\partial A_y}{\partial y}+A_x\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)=\frac{\partial}{\partial y}\left(\tfrac{1}{2}(A_x^2+A_y^2)\right)+A_x\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)\end{aligned}$$

These equations represent the vector $\mathbf{F}$ as the vector-sum of two vectors. The first is the ascendant of the scalar $\frac{1}{2}(A_x^2+A_y^2) = \frac{1}{2}A^2$. The second is the vector-product of the vectors $\text{curl}_z\mathbf{A}$ and $\mathbf{A}$. When we remember that the vector $\text{curl}_z\mathbf{A}$ is normal to the surface which contains $\mathbf{A}$, we see by the properties of the vector-product that this second vector will be directed along the positive normal to $\mathbf{A}$. Thus we can represent the scalar equation (d) by the vector-equation

$$(d')\qquad \mathbf{F} = \nabla(\tfrac{1}{2}A^2)+(\text{curl}_z\,\mathbf{A})\times\mathbf{A}$$

Thus we can also use the following method for constructing the field of the vector $\mathbf{F}$.

(1) We construct the scalar field of the half square of the intensity of the given vector (sec. 147), and then the field of the ascendant of this scalar (sec. 169).

(2) We construct the field of the curl of the given vector (section 172) and perform the graphical multiplication of this field by that of the intensity of the given vector. This field is considered as the intensity-field of a vector which has the direction of the positive normal to the given vector.

(3) We form the field of the sum of the two vectors, the fields of which we have found by the first two operations.

In most cases the first method will be preferable, as the two fields the vector-sum of which we shall form are then normal to each other. But still in special cases the second may be the shorter, for instance if some of the partial fields upon which the construction depends are already constructed for other purposes.

175. Pure Time-Differentiations and Time-Integrations of Scalar Fields.— While a pure space-differentiation is performed upon one chart, representing the field of a scalar or a vector at a given moment, the pure time-differentiations will consist in the comparison of two charts, which represent the field at two different moments.

Let a be a scalar which depends upon coordinates and time. Now let a_0 be the value of this scalar at a certain point at a time t_0, and a_1 its value at the *same point* at the time t_1. The quotient

(*a*) $$\bar{\varphi} = \frac{a_1 - a_0}{t_1 - t_0}$$

will then represent the *average* value which the differential-quotient

(*b*) $$\varphi = \frac{\partial a}{\partial t}$$

has at this point during the interval of time $t_1 - t_0$. If this interval is sufficiently short we can consider the value of the quotient (*a*) as identical with the value of the differential quotient (*b*) at the time

(*c*) $$t = t_0 + \frac{t_1 - t_0}{2}$$

If we know the field of the scalar a at two moments t_0 and t_1, which are separated by a sufficiently short interval of time $t_1 - t_0$, we can form the field of the derivative (*b*) at the time (*c*) in this manner:

We form by graphical subtraction the field of the difference

(*d*) $$a_1 - a_0$$

and afterwards perform the division of this field by the constant factor

(*e*) $$t_1 - t_0$$

The problem is thus reduced to algebraic problems which we have already treated. The only difficulty will be that the fields a_0 and a_1 may too closely resemble each other. Their equiscalar curves may cut each other under too small angles and it may be difficult to get a good drawing of that set of diagonal curves which represents the difference (*d*). It will be important to remark, however, that the errors will take precisely the same character as in the previous cases of differentiation: the curves representing the derivative will get an oscillating course, and these oscillations can be smoothed out afterwards. But in order to avoid these errors from the beginning, it will be important not to choose too short an interval of time (*e*). On the other hand it must not be chosen too long if it is to be allowed to identify, within the margin of allowable departures, the finite difference-quotient (*a*) with the differential-quotient (*b*) at the time (*c*).

The reversed problem, that of the pure time-integration, will be solved with the same ease. Let the field of a be given at the time t_0, $a = a_0$; and let the value of the derivative φ be known at any time t which is subject to the condition $t_0 < t < t_1$. If then the interval of time $t_1 - t_0$ is sufficiently short, we can identify the value of φ at the time t with the average value $\bar{\varphi}$ during the interval of time $t_1 - t_0$. We then find the value of a at the time t_1 by the formula

(*f*) $$a_1 = a_0 + \bar{\varphi}(t_1 - t_0)$$

Thus we have to perform the following graphical operations: first to multiply the field of the derivative $\bar{\varphi}$ by the interval of time $t_1 - t_0$, and then to perform the

graphical addition of the fields a_0 and $\bar{\varphi}(t_1-t_0)$. If we have sufficient knowledge of the derivative φ at different times t, we can repeat this operation and thus find at any time t the field a which is expressed analytically by the integral

$$(g) \qquad a = a_0 + \int_{t_0}^{t} \varphi \, dt$$

The graphical addition (f) will cause no such difficulty as that of the graphical subtraction (d). The only difficulty connected with the integration will arise from the gradual summing up of small errors from the one partial operation to the other.

176. Pure Time-Differentiations and Time-Integrations of Vector-Fields.— The principles for the pure time-differentiations will be precisely the same for a vector-field as for the scalar field.

Let **A** be a vector which depends upon both coordinates and time. Let it have the value $\mathbf{A}_0$ at a certain point at the time t_0, and the value $\mathbf{A}_1$ *at this same point* at the time t. The vector

$$(a) \qquad \overline{\mathbf{F}} = \frac{\mathbf{A}_1 - \mathbf{A}_0}{t_1 - t_0}$$

will then represent the average during the interval of time $t_1 - t_0$ of the vector

$$(b) \qquad \mathbf{F} = \frac{\partial \mathbf{A}}{\partial t}$$

which is the pure time-derivative of the vector **A** at the considered point. If we use sufficiently small intervals of time we can identify the vector $\overline{\mathbf{F}}$ with the value of **F** at the time

$$(c) \qquad t = t_0 + \frac{t_1 - t_0}{2}$$

By these formulæ we see at once that if we know the field of the given vector **A** at two moments t_0 and t_1, which are separated by a sufficiently small interval of time $t_1 - t_0$, we can form the field of the derivative at the time (c) in this manner:

We form the field of the vector-difference

$$(d) \qquad \mathbf{A}_1 - \mathbf{A}_0$$

and afterwards perform the division of this field with the constant factor

$$(e) \qquad t_1 - t_0$$

We have thus reduced the pure time differentiation of a vector-field to algebraic problems already treated. The only difficulty connected with this differentiation will consist in the formation of the vector-difference between two vector-fields which are very like each other. For this reason we must not choose too short an interval of time (e), just as we must not choose it too long if we are to be able to identify the two vectors (a) and (b).

For the formation of the vector-difference (d) we can use any of the methods which we have developed in vector-algebra. We can use the method of section 158 or the graphical tables (section 160), or finally the complete resultantometer (section 161). If we wish to use either of the first two methods, the field representing the difference of angle is first drawn as accurately as possible. The curves, as

they are obtained directly, will always have more or less of the oscillating course which is characteristic of curves obtained by a process of graphical differentiation. These oscillations should be carefully reduced. Then all results concerning singular points, etc., which can be obtained by use of the simple scalar addition or subtraction (section 159), must be worked out with the greatest care. The rest of the work

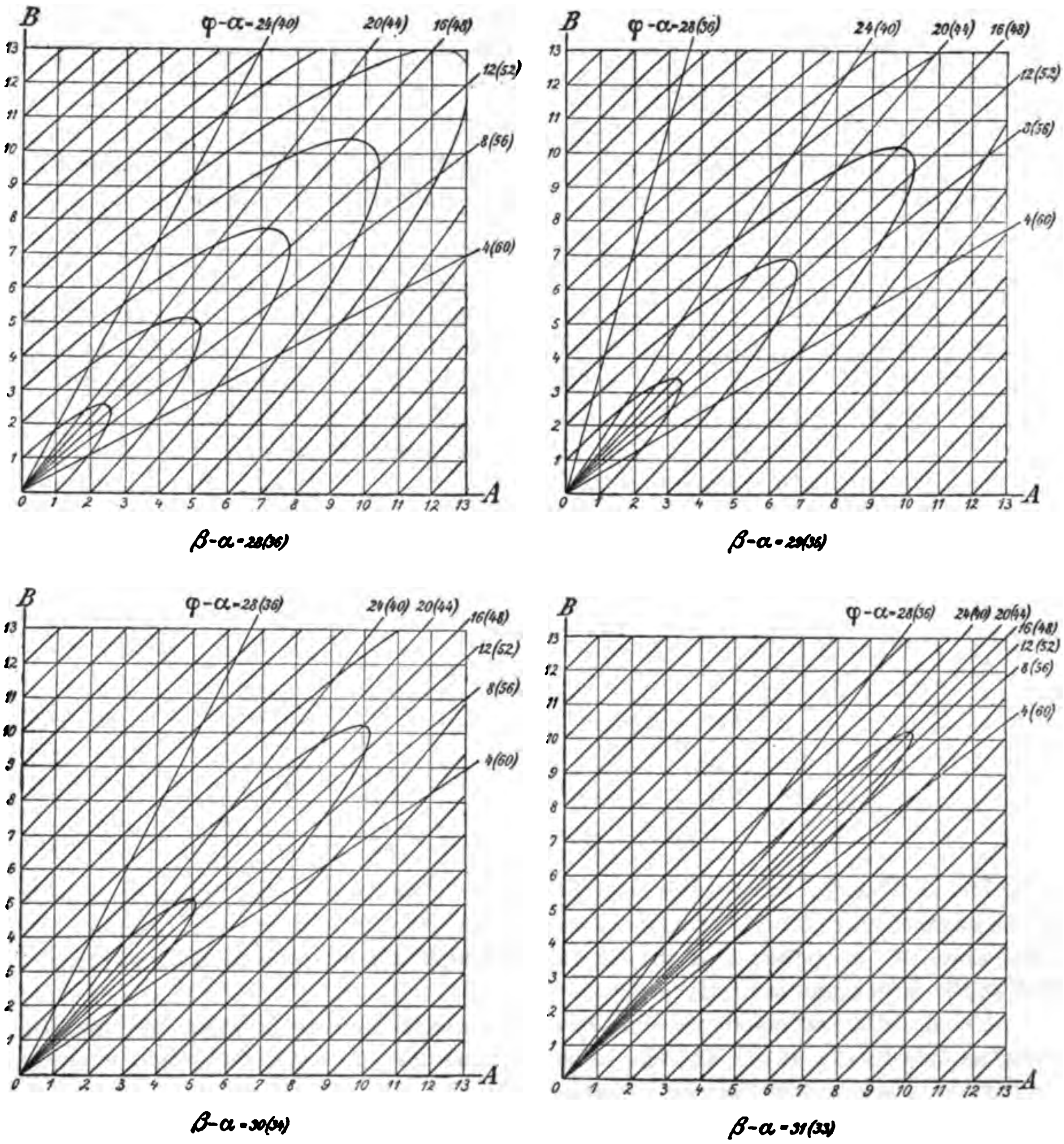


FIG. 101.—Graphical tables for time-differentiation of a vector-field.

will then mainly consist in forming the vector-sum of vectors which form angles differing very little from 32. If for this we wish to use graphical tables those of fig. 101 will serve the purpose. But in many cases the method of section 158 seems to be the best in spite of the greater number of separate operations.

When the field of the vector-derivative **F** is given at a series of epochs, and the field of the vector **A** at the initial epoch t_0, we can perform the pure time-integration, which is the inverse operation to the pure time-differentiation considered. We have then to identify the value of the derivative **F** at a moment t with the average derivative $\overline{\mathbf{F}}$ during a finite but short interval of time t_1-t_0 when $t_0<t<t_1$. We then perform the multiplication of the average derivative $\overline{\mathbf{F}}$ with the constant factor t_1-t_0, and afterwards perform the addition of the two vector-fields according to the formula

$$(f) \qquad \mathbf{A}_1=\mathbf{A}_0+\overline{\mathbf{F}}\,(t_1-t_0)$$

This operation may be repeated any number of times, and will lead to the field of the vector **A** at the time t, which is expressed analytically by the integral

$$(g) \qquad \mathbf{A}=\mathbf{A}_0+\int_{t_0}^{t}\mathbf{F}dt$$

The delicate point in this process of integration will be the addition of the generally very small vector $\overline{\mathbf{F}}(t_1-t_0)$ to the finite vector **A**. But as the isogons and the intensity-curves of the two fields will usually cut each other under finite angles, we shall not meet with the same difficulties as those connected with the differentiation. The only difficulty will be the gradual summing up of the small errors which enter at each partial operation.

177. **Complex Time and Space Differentiation.**—Besides the pure space-differentiations and the pure time-differentiations we shall also meet with complex space-time-differentiations. They will be seen to occur in all investigations concerning moving continuous media.

Let f be any function of coordinates and time,

$$(a) \qquad f(x, y, z, t)$$

It has four partial derivatives

$$(b) \qquad \frac{\partial f}{\partial x} \qquad \frac{\partial f}{\partial y} \qquad \frac{\partial f}{\partial z} \qquad \frac{\partial f}{\partial t}$$

The last is what we have called above the pure time-derivative. In order to form it we have to consider x, y, z as constant, and let only time vary; *i. e.*, we compare the values of f in the same locality at two different epochs. We shall therefore also call it the *local* time-derivative.

But on other occasions we shall have to compare the values which the function f has at two epochs at one and the same physical particle. What we keep constant in this comparison will then be not the locality x, y, z, in which the values of f are observed, but the *individuality* of the particle at which the values of f are observed. Now let v_x, v_y, v_z be the velocity-components of the particle. If at the time t it has the coordinates x, y, z, it will at the time $t+dt$ have the coordinates $x+v_xdt$, $y+v_ydt$, $z+v_zdt$. We have then to compare

$$(c) \qquad f(x+v_xdt,\ y+v_ydt,\ z+v_zdt,\ t+dt)$$

with $f(x, y, z, t)$. For this we can develop (c) according to Taylor's theorem, and leave quantities of the second order out of consideration. (c) then takes the form

$$f(x, y, z, t)+\frac{\partial f}{\partial x}v_x dt+\frac{\partial f}{\partial y}v_y dt+\frac{\partial f}{\partial z}v_z dt+\frac{\partial f}{\partial t}dt$$

The excess df of the value of f at the point $(x+v_x dt,\ y+v_y dt,\ z+v_z dt)$ at the time $t+dt$ over its value in the point (x, y, z) at the time t, will then be

$$df=\frac{\partial f}{\partial t}dt+\frac{\partial f}{\partial x}v_x dt+\frac{\partial f}{\partial y}v_y dt+\frac{\partial f}{\partial z}v_z dt$$

If we divide this equation by dt, we get a derivative which gives the rate of change of the value of f at one and the same moving material individuum. We shall call this the *individual* derivative, and denote it by $\frac{d}{dt}$. Its expression in terms of the four partial derivatives (b) will then be

$$\text{(d)}\qquad \frac{df}{dt}=\frac{\partial f}{\partial t}+v_x\frac{\partial f}{\partial x}+v_y\frac{\partial f}{\partial y}+v_z\frac{\partial f}{\partial z}$$

or in vector-notations

$$\text{(d')}\qquad \frac{df}{dt}=\frac{\partial f}{\partial t}+\mathbf{v}.\nabla f$$

A case of special importance is when f represents one component of a vector **A**. The individual time-derivative of the vector **A** will then be expressed by the three equations

$$\frac{dA_x}{dt}=\frac{\partial A_x}{\partial t}+v_x\frac{\partial A_x}{\partial x}+v_y\frac{\partial A_x}{\partial y}+v_z\frac{\partial A_x}{\partial z}$$

$$\frac{dA_y}{dt}=\frac{\partial A_y}{\partial t}+v_x\frac{\partial A_y}{\partial x}+v_y\frac{\partial A_y}{\partial y}+v_z\frac{\partial A_y}{\partial z}$$

$$\frac{dA_z}{dt}=\frac{\partial A_z}{\partial t}+v_x\frac{\partial A_z}{\partial x}+v_y\frac{\partial A_z}{\partial y}+v_z\frac{\partial A_z}{\partial z}$$

or, using the vector-notations introduced in section 174

$$\text{(e)}\qquad \frac{d\mathbf{A}}{dt}=\frac{\partial\mathbf{A}}{\partial t}+\mathbf{v}\nabla\mathbf{A}$$

An important case is that in which the vector **A** is the velocity of the moving particle. The rate of change of its velocity gives its *acceleration*, for which we thus get the equation

$$\text{(f)}\qquad \frac{d\mathbf{v}}{dt}=\frac{\partial\mathbf{v}}{t}+\mathbf{v}\nabla\mathbf{v}$$

In order to form the field of acceleration we have thus to perform pure time-derivations and pure space-derivations, which we have investigated already.

The distinction which we have here introduced between *local* and *individual* time-derivations will be of great importance in our continued work. The difference

between them can be very well illustrated in connection with the different methods of observing the meteorological elements. The instruments of the ordinary meteorological stations give the local variation of the meteorological elements. When we determine from the records of the barograph the rise of pressure per second, we get the local derivative of the pressure, $\frac{\partial p}{\partial t}$. In the same manner the thermograph of the station will give the local derivative of temperature $\frac{\partial \tau}{\partial t}$. By use of the windfane and the anemometer of the stations we can in the same way determine the local time-derivative of velocity $\frac{\partial \mathbf{v}}{\partial t}$. We may call this the *local acceleration*, to distinguish it carefully from the acceleration without further specification, which gives the rate of change of velocity of one and the same moving individuum.

Instead of considering the stationary instruments of a common station, we can now consider the moving instruments in a balloon, and let the balloon be in perfect equilibrium. It will then move along within one and the same mass of air. The barograph will then register the pressure of this mass of air, the thermograph will register its temperature. Forming from the records the rates of change, we get the individual time-derivatives $\frac{dp}{dt}, \frac{d\tau}{dt}$ . . . . If finally the velocity $\mathbf{v}$ of the balloon itself be registered, we should be able to determine the acceleration $\frac{d\mathbf{v}}{dt}$ of the mass of air in which it moves.

At the moment when this balloon with its moving instruments passes the station with its stationary instruments, the moving and the stationary instruments will show the same instantaneous values of the recorded quantities, but different rates of their change. Formula (*d*) will give the relation between the derivatives found from the records of the moving and the stationary instruments.

CHAPTER X.

THE FORCED VERTICAL MOTION AT THE BOUNDING SURFACES.

178. Hypsometric and Bathymetric Maps.—Having now developed the mathematical methods to be used, we can proceed to the accomplishment of the kinematic diagnosis. Chapters II–VII gave the direct methods for working out, from the observations, a complete diagnosis of the horizontal motion in atmosphere or hydrosphere. We shall now see how the correlated diagnosis of the vertical motion should be worked out.

The vertical motion begins at the bounding surfaces. Here the solenoidal surface-condition, section 115 (E), must be fulfilled; *i. e.*, both velocity and specific momentum must be tangential to the surface. The moving masses will be forced up or down according as the motion in horizontal projection goes against the slope or with it. We shall call the vertical motion which is produced in this way the "forced" vertical motion, to distinguish it from the "free" vertical motion to be considered in the next chapter.

In order to investigate this forced vertical motion, we must have complete topographic charts representing the configuration of the bounding surfaces; *i. e.*, we must have a complete representation of the topography of the world above as well as below sea-level. We have referred to such charts before, using them to define the spaces taken up by atmosphere and sea, and thus to give the extent of the fields representing the atmospheric or oceanic states. But the main influence which the bounding surfaces exert upon the internal structure of these fields comes through the forced vertical motion which arises as a consequence of the boundary condition. In view of this kinematic application we have worked out a representation of the topography of the world which is given on the first twenty-four sheets of the collection of plates which accompanies this work.

Our knowledge of the configuration of the bottom of the sea is still very incomplete; but fortunately most of the knowledge acquired has been made accessible by the bathymetrical map on a scale of 1 : 10 000 000 edited by the Prince of Monaco.* This map represents the topography of the earth below sea-level on 16 plates in Mercator and 8 in polar projection. We have for the main part copied our bathymetrical curves as well as the coast-lines from this chart, the most important changes being the following: Corrections and completion of the coast-lines in the Arctic and Antarctic regions have been performed according to the results of the well-known later Arctic and Antarctic expeditions. Changes in the course of the bathymetrical lines have been introduced, for the northern Atlantic according to Helland-Hansen and Nansen,† for the eastern Pacific according to the results

* Carte Générale Bathymétrique des Oceans, dressée par l'ordre de S. A. S. le Prince de Monaco.

† B. Helland-Hansen and Fridtjof Nansen: The Norwegian Sea. Christiania, 1909.

of the American *Albatross* Expedition* for different parts of the Indian Ocean and the western Pacific according to the results of the German *Planet* Expedition.†

While it has thus been easy to bring a bathymetrical chart representing in a tolerably satisfactory way our present knowledge of the configuration of the bottom of the sea, we have not been able to produce anything in the same manner satisfactory for the configuration of the crust of the earth above sea-level. The literature of cartography is remarkably poor as regards topographical charts of greater parts of the world. As it would have been quite impracticable for us to collect and utilize all primary material of topography in detail, which is accessible in the cartographical and geographical literature, we have chosen a limited number of sources. The most important of them has been the height numbers contained on Stieler's Atlas‡ used in connection with the course of the rivers and the shadings representing the orographical features of the countries. Besides these we have used a map of the world on a smaller scale edited by the German Marine Authorities,§ which contains the height-curves for 300, 1000, and 2000 meters. During our work Romer's Atlas|| appeared, containing on a small scale charts of the continents, with height-curves corresponding to the interval of 1000 meters. The topography for the United States has been taken from the chart of the Geological Survey, the curves being changed from feet to meters. Special attention has been paid to the latest results of Sven Hedin in Central Asia.¶ The short pieces of height-curves drawn on the chart of the Antarctic continent are derived from Shackleton's chart.** For the drawing of the height-curves in the Arctic regions, we are indebted to Nansen, Isaachsen, and Amundsen for valuable hints.

The chart which we have thus produced must not be considered as a geographical document, and it is to be hoped that better charts may soon be produced by professional geographers. But it will serve our special purposes very well.

Our chart is on the scale of 1 : 20 000 000, and like that of the Prince of Monaco it is distributed on 16 plates in Mercator's projection and 8 in polar projection. It gives the height above and the depths below sea-level precisely in the same way. The curves for the height, respectively the depth, of 200 meters are dotted, those for 500 stippled, and then continuous curves are drawn for every 1000 meters of height or depth. It will be equally legitimate to interpret the meter indicating these heights or depths as the common geometrical meter or as the dynamic meter (compare Statics, section 15).

179. Charts of Idealized Topography.—If we were to proceed with perfect rigor, we should have to apply the surface-condition to the true surface of separation between the moving medium and the bounding surface. This would require the

*Memoires of the Museum of Comparative Zoology at Harvard College, vol. 33. Cambridge, 1906.

†Forschungreise S. M. S. *Planet* 1906–1907. T. 3, Oceanographie. Berlin, 1909.

‡Stieler's Hand-Atlas, Neunte Auflage. Gotha, 1907.

§Weltkarte zur Uebersicht der Meerestiefen & Höhenschichten, herausgegeben von dem Hydrographischen Amte des Reichs-Marine-Amts. Berlin, 1893.

||Lemberg, 1908.

¶Sven Hedin: Transhimalaya. Stockholm, 1909.

**B. E. H. Shackleton: The Heart of the Antarctic. London, 1908.

construction of topographic maps of a completeness which can not be attained Taking the case of the atmosphere, the chart should give the configuration of every irregularity of the ground, every stone, every tree, every house. And the use of the map would require wind-observations taken all around these irregularities.

Just as we have been obliged to consider an idealized wind (section 97), we must use an idealized topography, corresponding to the placing of the fanes and the anemometers in open places, above that sheet of air which has the most irregular motions.

It will therefore be perfectly legitimate to use an idealized topography like that which is represented by the common contour-lines. And in most cases it will be not only legitimate, but necessary, to go still further in the idealization than on common charts. Even the map of the world as we have drawn it on the plates I–XXIV contains far too much detail for meteorological work as long as the phenomena are to be studied on a large scale, and not in minute details.

For our practical work we have therefore been obliged to derive from this map *special maps of idealized topography*. All these special maps have been drawn on a scale of 1 : 10 000 000. We have found this scale convenient for the performance of our constructions, and all our graphical auxiliaries have been made with this scale in view. All these special maps have been drawn in a conical projection corresponding to the latitude. Our reasons for preferring this projection to one with curved meridians have been given already; all kinds of auxiliary graphical instruments (sections 143, 161, 163) are easily applied when the chart is in conical projection. The idealizations have been performed step by step. First we have drawn a map where all the smallest irregularities of the contour-lines have been removed, then a new map where greater irregularities have been removed, and so on. The simplified curves are always drawn so that the volumes of the great mountain-chains and of the continents have retained their value. In this manner correct values will be found for the average intensity of the forced ascending or descending motion, while the small irregular motions up and down, which are only of local importance, will drop out. But it should be remembered that the drawing of the idealized charts has no unique solution. The same degree of idealization can be attained in different ways as regards details. It will be a question of experience to find out the proper degree of idealization and the best solution of dubious questions of detail. In practical work we have used two degrees of idealization, represented by the "moderately idealized" charts of the United States and of Europe given on plates XXV and XXVIII, and the "greatly idealized" charts of plates XXVI and XXIX. We have used the moderately idealized charts more for qualitative purposes, drawing on them the charts of the horizontal motion (section 135), while we use the charts of greatly idealized topography for the rigorous quantitative work.

We have given no examples of idealized bathymetric maps. As we have had no observations from which we could work out a kinematic diagnosis of sea-motions, we have had no opportunity of examining the question of such charts for hydrographic purposes. It should be remembered, however, that the bottom of the sea is, generally speaking, less irregular than the ground above sea-level, and at the same

time our knowledge is less detailed. When in spite of this further idealizations have to be performed, great care should be taken, for small irregularities of the bottom may influence the motion of the sea much more than corresponding irregularities of the ground are able to influence the motion of the air.*

When in the following we speak of the ground, we always mean the ideal surface which is represented by our charts. We shall consider the wind-observations obtained at the meteorological stations as representing the air-motion at this surface itself. This will be perfectly legitimate from a kinematic point of view. But the real removing of all irregularities would of course have great dynamic consequences. We shall therefore be obliged later to consider this ideal surface as offering a frictional resistance which a smooth surface would not offer in reality.

180. The Motion in the Lowest Surface of Flow.—The particles of the moving medium which are in contact with the bounding surface will move tangential to it in virtue of the solenoidal surface-condition. Therefore a hypsometric map represents directly the topography of the lowest surface of flow in the atmosphere; and in the same manner a bathymetric map represents the topography of the lowest surface of flow in the sea.

When we shall represent the motion in this lowest surface of flow, we must remember its exceedingly minute inclination. Even on our charts of moderately idealized topography hardly any place will be found where contour-lines corresponding to a difference of level of 1000 meters approach each other as closely as 1 mm. On a chart on a scale of 1 in ten millions, this will give an inclination which is smaller than one in ten. The cosine of the angle of inclination will therefore be greater than 0.995, and when we set this cosine equal to unity, we shall never make errors as great as 0.5 per cent. Such errors will be insignificant compared with the errors of observation. *We need therefore make no difference between the numerical values of the horizontal component of the motion and the resultant motion itself which is parallel to the ground.*

For this reason we shall get a representation of the motion along the bounding surface simply by drawing the lines of flow and the curves of equal intensity on outline-maps which contain the contour-lines. The three sets of lines, contour-lines, lines of flow (respective isogonal curves), and intensity-curves give a complete representation of the surface of flow and of the motion in it (compare fig. 43 A and fig. 45 A).

181. Charts of Vertical Velocity at the Ground.—From a chart containing these three sets of lines we can easily draw a special chart of the vertical component of the motion. When s is a line of flow in the atmosphere and z its height above sea-level its angle of inclination will be

$$i = \frac{dz}{ds} \tag{a}$$

*Cf. the notes, pp. 58 and 59.

v being the resultant velocity, the vertical component v_z will then be given by the formula

$$(b) \qquad v_z = v\frac{dz}{ds}$$

In accordance with this expression we can construct the field of v_z. In the case of motion along the bottom of the sea we should have to use the depth below sea-level instead of the height above it. But we shall henceforth consider exclusively the case of the atmosphere. As soon as the observations are at hand, it will be easy to adapt the same methods to the investigation of sea-motions.

Formula (b) reduces the drawing of a chart of vertical velocity to a simple problem of graphical differentiation and of graphical algebra.

A rough sketch of the field (b) can easily be made by the discontinuous method. Evidently the field (b) will contain a zero-line $v_z = 0$, which separates from each other the windward and the leeward sides of the mountains. The general course of this line is seen at once and can be drawn by eye-measure in those parts of the country where the slope is strong enough to produce a vertical motion of any importance. By use of the differentiating sheet of fig. 81, we can then make a few determinations of v_z in the places where it is seen to have its greatest positive and negative values. Afterwards the curves $v_z = \text{const.}$ can be drawn by eye-measure. It will not be difficult in this way to draw such charts in the daily meteorological service.

For more detailed investigations we can bring the continuous graphical methods into application. The method of proceeding will be this:

We construct first the chart of the angle of inclination (a). The construction is that which has been exemplified in fig. 83. In this figure we can interpret the lines α = const. as contour-lines, and the lines s as the lines of flow of the wind. The stippled curves will then be curves for equal values of the angle of inclination i. Of these curves we first draw that for the angle of inclination zero. This curve will pass through all the points of tangency of the lines of flow and the contour-lines. A zero-curve must therefore pass the summit of every mountain as well as the highest point in every pass. Inasmuch as the wind does not travel precisely along the chain, but has a component across it, the zero-line will follow near the highest ridge of the chain, passing all the summits and the highest point of the passes. In the same manner, when the wind does not travel precisely along a valley, but has a component across it, a zero-line will run along it, near its bottom.

As soon as the zero-line is drawn, we determine the course of the curves for integer values of the angle of inclination by making continuous use of the differentiating sheet of fig. 81 as described in section 165.

Finally we perform the graphical multiplication (section 150) of the field of the angle of inclination i with that of the scalar value v of the velocity of the wind. The chart resulting will then represent the field of the vertical velocity v_z.

182. Ascendant-Charts.—From a theoretical point of view the drawing of the charts of vertical velocity is exceedingly simple. But still, when it is to be done

with care for the details, it will prove to be the most laborious operation of kinematic diagnosis. The reason is that in spite of all idealizations, the topographic chart will remain more complicated than the charts which represent the field of the meteorological quantities observed.

In order to simplify the work another way may be suggested: From the topographical map we could derive a chart representing the ascendant of the ground, and print it as a blank. The process of differentiation would then be performed once for all; for it is easily seen that the vertical velocity may be expressed as the scalar product of this ascendant and the horizontal velocity. Each chart of vertical velocity could then be derived by a simple algebraic process (section 156). But when this method does not work as well as might be expected, it is due to the great complexity of the isogons and the intensity-curves representing the ascendant. The control due to direct intuition is lost, and keen attention will be required to avoid mistakes; but this method may be considered if extensive detailed investigations on the vertical motion at the ground are to be performed.

A method which also might be considered in such a case would be the consistent use of rectangular components. If the W.–E. and the S.–N. components of the wind were observed, we might draw and print as blanks two special auxiliary charts, one of the W.–E. component and one of the S.–N. component of the ascendant. By a simple graphical multiplication we should then be able to derive a chart of the vertical velocity due to each component of the wind, and afterwards a chart of the total vertical velocity by graphical addition.

183. Change of Velocity into Specific Momentum. Charts of Density at the Ground.—If we have a chart which represents the density of the air at the ground, we can at once by graphical multiplication change a chart of velocity into one of specific momentum. It will be sufficient if the chart of density has an accuracy corresponding to that of the wind-observations. We can then ignore the influence of humidity on density and consider density as a function only of pressure and temperature. When we know the topography of the isobaric surfaces in free space, we can draw their curves of intersection with the ground. These curves will give a chart of the pressure at the ground. By this chart, together with a corresponding chart of temperature at the ground, we can draw a chart of the density at the ground, using one of the two auxiliary tables N.

Table N, A, contains density and pressure as argument, and temperature as the tabulated quantity. It gives the temperature of the point where the equiscalar curves for the required field of density cut the given isobaric curves. Table N, B, contains density and temperature as arguments and gives the pressure of the points where the required curves for equal density cut the given isothermal curves.

A density-chart drawn by one of these tables will possess an accuracy far exceeding that of the observations of velocity. In most cases we can therefore still further simplify the method, treating pressure at the ground as if it depended only upon the height above sea-level and ignoring its variations from day to day. We can then get the density of the air as function of height and temperature.

When we use the average relation between pressure and height given in Statics, table A, p. 29, we get the tables O. The first gives the temperature at points where the contour-lines of the topographic map are cut by the required curves of equal density, while the second gives the height of the points where the required density-curves cut the given isotherms. As density is proportional to pressure, and as pressure at a place will as a rule differ only a small percentage from its average

TABLES N.

A. *Temperature at points where isopycnic curves cut isobaric curves.*

ρ	p								
	1100	1050	1000	950	900	850	800	750	700
0.0008								54	32
09						57	37	18	−1
10				58	40	26	6	−12	−29
11			44	28	12	−4	−20	−36	−51
12	46	32	17	3	−11	−26	−40	−55	
13	22	8	−5	−18	−32	−45	−59		
14	1	−12	−24	−36	−49				
15	−18	−29	−36	−53					

B. *Pressure at points where isopycnic curves cut isothermic curves.*

ρ	t								
	−40	−30	−20	−10	0	10	20	30	40
0.0008				604	627	650	673	696	719
09			653	679	705	731	757	783	808
10	669	697	726	755	784	812	841	870	898
11	736	767	799	830	862	893	925	957	988
12	802	837	871	906	940	975	1009	1044	1078
13	869	907	944	981	1019	1056	1093		
14	936	976	1017	1057	1097				
15	1003	1046	1089						

value, the use of this simpler method will generally give only a small percentage of errors of the values of density. This will be of no importance when the general inaccuracy of the observations of velocity is considered.

Instead of drawing a chart of temperature at the ground, we can take an average value of the temperature, and draw the density-chart by use of that column in table O, B, which corresponds to this temperature.

TABLES O.

A. *Temperature at points where isopycnic curves cut contour-lines.*

ρ	z					
	0	200	500	1000	2000	3000
0.0008						25
09					31	−7
10			54	35	−2	−34
11	44	36	25	7	−27	−47
12	17	10	0	−16	−46	
13	−5	−11	−21	−36		
14	−24	−30	−39	−53		
15	−41	−47	−56			

B. *Height of points where isopycnic curves cut isotherms.*

ρ	t								
	−40	−30	−20	−10	0	10	20	30	40
0.0008						3400	3100	2900	2600
09			3400	3100	2800	2500	2200	2000	1700
10	3200	2900	2600	2300	2000	1700	1400	1100	900
11	2400	2100	1800	1500	1200	900	600	400	100
12	1800	1400	1100	800	500	200			
13	1100	800	500	200					
14	500	200							
15									

We can even simplify still further, and neglect also the variations of temperature. We then consider density as given only by the height above sea-level, for instance, by the column for 0° C of table O, B.

In this case we should then always use the same density-chart, which could be derived from the contour-lines of the topographic chart by use of the numbers in this column.

In all cases when the density-chart is found, we have simply to perform the graphical multiplication of the charts of vertical velocity by that of density in order to get the chart of vertical specific momentum.

184. Direct Method of Determining Vertical Specific Momentum from Horizontal Velocity at the Ground.—If the chart of vertical velocity is drawn already, the method given in the preceding section will give the easiest construction of the chart of vertical specific momentum. But we can also use a direct method without passing through the vertical velocity. As the contour-lines of our charts can be interpreted as lines of equal dynamic height H, we can write equation 181 (b)

$$(a) \qquad v_z = v\frac{dH}{ds}$$

Now, according to the fundamental equation of hydrostatics, we have $dH = -\alpha dp$, where pressure p is measured in decibars when dynamic height H is measured in dynamic meters. When we introduce this and divide by the specific volume α, *i. e.*, multiply by the density ρ, we get on the left side the vertical component V_z of specific momentum,

$$(b) \qquad V_z = -v\frac{dp}{ds}$$

This equation gives the following rule for drawing the chart of vertical specific momentum at the ground: We first draw the chart which represents the field of pressure at the ground; then we perform the graphical differentiation of this field with respect to the length of arc along the lines of flow; finally we perform graphical multiplication of the field thus obtained by the field of the scalar value v of velocity.

This method is precisely like that which we have developed for the velocity except that we use the chart of pressure at the ground instead of the topographic chart. But it will give more work, inasmuch as the topographic map always remains the same, while that of pressure changes and must be drawn again in each case. If we ignore, however, the variations in time of the pressure, we can draw a chart representing the average pressure at the ground and use this chart consistently for the determination of vertical specific momentum, precisely as the topographic map for the determination of the vertical velocity. Then it will be as easy to draw charts of vertical specific momentum as of vertical velocity. The errors in the determination of vertical specific momentum caused by the use of the average pressure will amount to a small percentage and thus always be small compared to those which arise from the imperfectness of the observations of the wind. Therefore in general there will be no objection to using this simplified method.

We have therefore drawn the charts of plates XXVII and XXX, which give the average pressure at the ground in the United States and in Europe. As to the degree of idealization, they correspond to the strongly idealized topographic maps of plates XXVI and XXIX. The coast-line is to be considered as an isobaric line of pressure about 1013 m-bar. Then the curves for 1000, 900, 800, . . . m-bar have been drawn as continuous lines, while a curve for the pressure of 980 m-bar is dotted and a curve for 950 m-bar is stippled.

CHAPTER XI.

VERTICAL MOTION IN FREE SPACE—COMPLETE KINEMATIC DIAGNOSIS.

185. Free Vertical Motion.—As the distance from the bounding surface increases, the forced vertical motion produced at this surface will gradually be modified. An additional vertical motion will arise in the free space and conjoin with the forced vertical motion. We shall for the sake of brevity call it the *free* vertical motion. It can be investigated by the solenoidal condition in space, precisely as the forced vertical motion by the solenoidal surface-condition.

We have done it already from a qualitative point of view (Chapter V). We had to take the free vertical motion into consideration in order to explain the features of the horizontal motion. The vertical motion existing above centers or lines of convergence and of divergence gives typical examples of this free vertical motion and shows its connection with the horizontal motion. It will therefore be understood at once that from a given horizontal motion we can derive the correlated vertical motion by making quantitative use of the solenoidal condition.

The vector which fulfils the solenoidal condition with the highest degree of approximation is specific momentum. Both in atmosphere and in hydrosphere the field of mass can be considered as stationary in space (section 117). Therefore the mass-transport leading into a stationary volume through one part of the bounding surface will be equal to that leading out of it through other parts of this surface. The solenoidal nature of specific momentum is a consequence of this property of the mass-transport. In the hydrosphere the moving masses can be considered as incompressible. Then the volume-transport obtains the same property as the mass-transport, and even velocity will be a solenoidal vector. But in developing our methods we shall consider only atmospheric motions. Their adaptation to sea-motions will cause no difficulty as soon as the observations to be used are at hand.

186. Diagnostic Use of the Solenoidal Condition.—We shall consider an atmospheric sheet limited by two horizontal or quasi-horizontal surfaces. dz will be their vertical distance. The average horizontal motion in this sheet will be represented by the specific momentum $\mathbf{V}$. A chart will be given containing the lines of flow (or the isogons) and curves for equal intensity $V =$ const. of this vector. By using the solenoidal condition we shall derive from this chart the correlated data regarding the vertical motion. We will give three different methods of deriving these data, each leading to a special form for the representation of the vertical motion.

(A) *Areas of equal vertical transport.*—The simplest plan will be to draw a chart of the horizontal transport T in the sheet. By the solenoidal condition this chart must necessarily give an indirect representation also of the correlated vertical

transport T_v. Let dn be a horizontal element of line which is normal to the lines of flow. The expression

$$(a) \qquad T = Vdndz$$

will then give the horizontal transport through the area $dndz$, which extends from the bottom to the top of the sheet. Thus we have to draw a chart representing the expression (*a*).

In order to do this we shall first consider the expression

$$(b) \qquad T_1 = Vdn$$

which represents the transport in a sheet of the thickness of $dz = 1$. The curves T_1 = const. will be the curves of equal transport for the two-dimensional vector **V**. In order to draw these curves we may proceed as we have developed already (section 167): On the chart which represents **V** we first draw an arbitrary initial curve C' and divide it into elements which give equal values of the two-dimensional transport; *i. e.*, for each element we shall have

$$(c) \qquad V'dn' = c'$$

dn' denoting the projection of the element of the curve C' upon the normal to the lines of flow. c' is an arbitrarily chosen constant, equal either to the unit of transport used in practice or equal to a simple multiple or fraction of this unit. The essential point is to choose the constant so that we get bands of flow of suitable breadth for the construction. Through the points of division we draw lines of flow which will then define the bands of flow to which the transport T is to be referred. Using the divided sheet of fig. 86, we then draw curves for equal values of the breadth dn of these bands of flow. Finally we perform the graphical multiplication of this field by that of V. The field resulting will be that of T_1, which represents the horizontal transport in a sheet of unit thickness, $dz = 1$.

In order to get a chart of T we have finally to perform the multiplication by the thickness dz of the sheet. If dz is constant this will lead to a simple change of the intervals between the curves T_1 = const. In the general case, where the thickness of the sheet is variable from place to place, dz will be represented by a chart which gives the topography of the upper limiting surface of the sheet relatively to the lower. We have then to perform the graphical multiplication of this field by that of T_1. The result will be the field of T represented by curves for integer values

$$T = \ldots 11, 10, 9, 8, \ldots$$

This field directly represents the average horizontal transport in the sheet, but indirectly it will also represent the correlated free vertical transport. Let us suppose, for the sake of simplicity, that the lower limiting surface of the sheet is a surface of flow. The bands of flow in the two-dimensional drawing will then represent tubes, the bottom and the two lateral walls of which are surfaces of flow, while a transport goes through the top. The curves T = const. will represent vertical walls which are sections of these tubes. When we proceed along a tube

from one section to the next, we have unit change of horizontal transport. By the solenoidal condition we must therefore have unit vertical transport through that area of the top which is contained between these two sections. Thus the curves T = const. will divide the bands of flow into areas for each of which we have unit vertical transport through the upper limiting surface of the sheet. In the case of decreasing horizontal transport the vertical transport will go up, and in case of increasing vertical transport it will go down through the top of the sheet.

If there is a vertical transport through the lower limiting surface of the sheet, the areas will represent that addition to the vertical transport which arises on account of the horizontal motion in the sheet.

We thus see that we have a method of arriving at a representation of vertical motion like that illustrated by figs. 43 c and 45 c.

(B) *Topographic method.*—We shall retain that division of the given chart of V into bands of flow which we have performed as an introduction to the construction of areas of equal vertical transport. The curve C' represents a vertical wall of the given constant height dz'. The bands of flow on the chart represent tubes of flow in space, which at this wall have the given transport $T' = V'dn'dz'$. In case (A) we have examined the change of transport T as we proceeded along tubes, which were limited below and above by given surfaces. Now only the lower limiting surface will be given. The upper will be subject to this condition, that it shall pass through the upper edge of the wall C'. We will determine its height dz above the lower surface so that the tubes retain in all sections the transport T' which they have in the section formed by the wall C'.

For this we have to introduce into (a) the value $V'dn'dz'$ for T, and to solve with respect to dz,

$$dz = \frac{V'dn'}{Vdn}\,dz' \tag{d}$$

and construct a chart of this height dz. This will be a topographic chart which gives the height of the upper limiting surface relatively to the given lower surface.

The construction will be very like the preceding one. We first perform the construction for the case of a wall C' of unit height. Setting $dz' = 1$ and remembering that $V'dn'$ has been determined to be equal to the number c', we have

$$dz_1 = \frac{c'}{Vdn} \tag{e}$$

As c' is equal either to unity or to a simple multiple or decimal fraction of the unity, we can determine the field of the quantity $\frac{c'}{dn}$ in one operation, using the divided sheet of fig. 81. Then we perform the graphical division of this field by that of V. The field resulting will be a topographic chart representing the upper limiting surface when the initial wall C' has unit height.

Performing the multiplication by the constant height dz' we get the field of dz, *i. e.*, the topographic chart representing the upper limiting surface for any given constant height of the initial wall C'.

The interpretation of the chart will be easiest in the case where the given lower limiting surface of the sheet is a surface of flow. The transport in each tube being constant, we conclude by the solenoidal condition that the upper limiting surface will also be a surface of flow.

We have thus obtained a method of constructing the topography of one surface of flow relatively to another, and thus of arriving at those representations of vertical motions which are illustrated by the figures 43 A and B and 45 A and B.

If the lower limiting surface of the sheet is not a surface of flow, the upper surface (the topography of which we have determined) will not be one either. But still it will characterize that part of the vertical motion which arises as a consequence of the horizontal motion within the sheet.

(C) *Vertical component of specific momentum.*—If we wish to find the vertical component of specific momentum, we have simply to use the solenoidal condition in its differential form. By equation (*j*) of section 171, we have

$$\frac{\partial V_z}{\partial z} = -\mathrm{div}_2\mathbf{V} \tag{f}$$

or, when we multiply by dz,

$$dV_z = (-\mathrm{div}_2\mathbf{V})dz \tag{g}$$

By this equation we can draw a chart of the increase dV_z of vertical specific momentum within a sheet of any thickness dz within which we know the horizontal specific momentum $\mathbf{V}$.

As in the preceding cases, it will be convenient to begin with the case of a sheet of unit thickness $dz = 1$. The corresponding increase of vertical specific momentum will be

$$dV_{1,z} = -\mathrm{div}_2\mathbf{V} \tag{h}$$

From the given chart which represents the field of the horizontal vector $\mathbf{V}$ we derive the field of the divergence $\mathrm{div}_2\mathbf{V}$, using the method developed in section 170. This field of divergence will, after change of sign, represent the increase $dV_{1,z}$ of vertical specific momentum from bottom to top in a sheet of unit thickness.

In order to get the increase dV_z for a sheet of any thickness we have to perform the multiplication by the thickness of dz. If dz is constant, this will simply be a change of the interval between the curves for constant values of $dV_{1,z}$. In the general case where dz is variable, and is represented by a chart which gives the topography of the upper limiting surface of the sheet relatively to the lower, we have to perform the graphical multiplication of the fields of $dV_{1,z}$ and of dz.

187. Change of Variables.—The horizontal mass-transport was given by the formula

$$T = Vdndz$$

It is the dz appearing here which brings in the vertical dimension in the formulæ of the preceding section and allows us to describe the motion in reference to this dimension.

Instead of expressing the vertical dimension in the direct way by the length dz measured along a vertical line, we can express it indirectly by the decrease of pressure $-dp$ along this line. For when the field of pressure is known, the indication of a pressure will be equivalent to that of a height. In order to bring in pressure we can first substitute dynamic height H for geometric height z. This can be done with sufficient accuracy by the relation

$$dz = 1.02\,dH$$

dz being expressed in meters and dH in dynamic meters. Then we can pass from dynamic height to pressure by the equation of hydrostatics

$$dH = -\alpha dp$$

where pressure p is to be expressed in decibars and H in dynamic meters. When we introduce this in the expression of T and remember

$$v = \alpha V$$

we shall get as a new expression of the horizontal mass-transport

$$(a) \qquad T = (1.02\,v)\,dn\,(-dp)$$

or, when we leave out the practically insignificant factor 1.02

$$(a') \qquad T = v\,dn\,(-dp)$$

When we compare this expression with the original, $T = Vdndz$, we conclude that in the formulæ of the preceding section we are entitled to introduce the decrease of pressure $-dp$ instead of the increase of height dz on condition of introducing at the same time horizontal velocity v instead of horizontal specific momentum V. This change of the formulæ leads at once to the following general rule:

The constructions described in the preceding section may be performed upon charts of horizontal velocity **v** *instead of upon charts of horizontal specific momentum* **V**. *The charts resulting will then describe the vertical motion in reference to the pressure decreasing upward instead of in reference to the height increasing upward.*

Thus to mention the special cases:

(A) *Areas for equal vertical mass-transport.*—We start with a chart representing horizontal velocity, and propose to draw a chart representing the transport (a').

For this we first draw a chart of the expression

$$(b) \qquad T_1 = vdn$$

which represents the horizontal mass-transport in a sheet of a thickness defined by unit decrease of pressure from bottom to top, $-dp = 1$. In order to get this chart we first draw an initial curve C' and divide it into elements which give

$$(c) \qquad v'dn' = c'$$

where dn' denotes the projection of the element of the curve C' on the normal to the lines of flow, and c' is a constant chosen so as to get proper breadths of the bands of flow. Through the points of division we draw lines of flow dividing the field into the bands of flow to which the transport T_1 is to be referred. Then we draw curves

for equal values of the breadths dn of these bands of flow and perform the graphical multiplication of this field by that of the scalar value v of the velocity. This gives the field of T_1.

The field of T_1 will represent the final result if the thickness of the sheet is defined by unit decrease of pressure. If it has a thickness defined by any variable decrease of pressure, a chart of this decrease of pressure $-dp$ must be given.

This chart will give in terms of pressure the topography of the upper limiting surface of the sheet relatively to the lower one. If we perform the graphical multiplication of this field of pressure $-dp$ by that of T_1, we get the field of T.

The direct interpretation of the chart of T is this: it gives the *horizontal* mass-transport in the sheet the thickness of which is defined by the decrease of pressure $-dp$ from bottom to top. But at the same time it represents the *vertical* mass-transport through the top of this sheet in an indirect way: The curves $T=$const. divide the bands of flow into elementary areas; for each of these areas we have unit mass-transport through the upper limiting surface of the sheet.

(B) *Topographic method.*—We retain that division of the given velocity-chart into bands of flow which we have performed by drawing the curve C' and dividing it into elements. The curve C' will now represent a vertical wall the height of which is given by the condition that there shall be constant decrease of pressure $-dp'$ from bottom to top. At this wall the tubes will then have the given mass-transport $T' = v'dn'(-dp')$. We propose to draw a chart of that decrease of pressure

$$(d) \qquad -dp = \frac{v'dn'}{vdn}(-dp')$$

which must define the thickness of the sheet if the tubes are to have everywhere the same mass-transport as they have at the wall C'.

We perform the construction first for the case in which the wall C' has the height which is defined by unit decrease of pressure from bottom to top, $-dp'=1$. This is done according to the formula

$$(e) \qquad -dp_1 = \frac{c'}{vdn}$$

where c' is the value of the two-dimensional transport $v'dn'$ at the curve C'. In order to find the field of $-dp_1$ we first draw the field of $\frac{c'}{dn}$ by use of the differentiating sheet of fig. 81. Then we perform the graphical division by the field of the scalar value of the velocity v. The resulting field will be a chart which gives in terms of pressure the topography of the upper limiting surface of the sheet relatively to the lower one in the case $-dp_1 = 1$. If the wall C' has a height defined by another constant decrease of pressure $-dp'$, we have finally to perform the multiplication of the field of $-dp_1$ by this constant $-dp'$. The field resulting (d) represents in terms of pressure the topography of the upper limiting surface of the sheet relatively to the lower one. If the lower is a surface of flow, the upper will also be a surface of

flow in virtue of the solenoidal condition. We thus have a method of drawing charts of surfaces of flow in the atmosphere, giving the topography of these surfaces in reference to the field of pressure.

(C) *Vertical component of specific momentum.*—When we make the change of variables in the solenoidal condition in its differential form we shall come to the equation

$$\frac{\partial V_z}{-\partial p} = -\mathrm{div}_2\mathbf{v} \tag{f}$$

or, solving with respect to the increase dV_z of vertical specific momentum, we get

$$dV_z = -(\mathrm{div}_2\mathbf{v})\ (-dp) \tag{g}$$

By use of this equation we can find the increase dV_z of vertical specific momentum in a sheet the thickness of which is defined by the decrease of pressure $-dp$.

The practical work will begin by drawing a chart for the case in which the sheet is defined by unit decrease of pressure, $-dp=1$. The increase of vertical specific momentum in this sheet will be

$$dV_{1,z} = -\mathrm{div}_2\mathbf{v} \tag{h}$$

That is, it will be found if we draw the field of divergence of the given field of horizontal velocity $\mathbf{v}$, and then change the sign.

From a sheet defined by unit decrease of pressure we can pass to one for any decrease of pressure by multiplication by that pressure $-dp$ which defines the thickness of the sheet. If $-dp$ is constant, the result will simply be a change of the interval between the curves which represent $dV_{1,z}$. If $-dp$ is variable from place to place, it must be represented by a chart, which will then represent the topography of the upper limiting surface of the sheet relatively to the lower one, topography being expressed by decreases of pressure instead of by increases of height. By graphical multiplication of the chart of $-dp$ by that of $dV_{1,z}$ we shall then arrive at the chart of dV_z, which represents the increase of vertical specific momentum in a sheet of any variable thickness.

188. Example. Cyclonic Center, United States of America, November 28, 1905.—As the two sets of parallel methods which we have developed in the two preceding sections lead to precisely the same formal constructions, it will be sufficient to exemplify one of these sets. We shall take that of section 187, as we can then apply directly the chart of observed horizontal velocity without changing it first into a chart of specific momentum.

In all cases we have to start with the chart of fig. 102, which represents the observed horizontal velocity at 8 a. m., 75th meridian time. The fine lines are curves for equal wind-velocity, expressed in meters per second. The thick lines with arrow-heads are the lines of flow, which are seen to run into a marked center of convergence. For further data regarding the meteorological conditions at the epoch of observation see plates XXXV and XXXVI.

The chart of fig. 102 is on a scale of 1 : 10 000 000. Thus 1 centimeter on the chart represents 100,000 meters. As the centimeter is the unit length on our divided sheets, we see that by using them for measurements on our charts we express horizontal distances in a unit length of 10^5 meters.

(A) *Areas of unit vertical transport.*—We draw the curve C', fig. 103, and divide it into elements which give $v'dn' = 5$. (The value $v'dn' = 1$ would have given too narrow bands for a good construction.) Through the points of division we draw new lines of flow which define the bands of flow to which the transport shall be referred. On the chart which represents these bands we have also copied the curves of

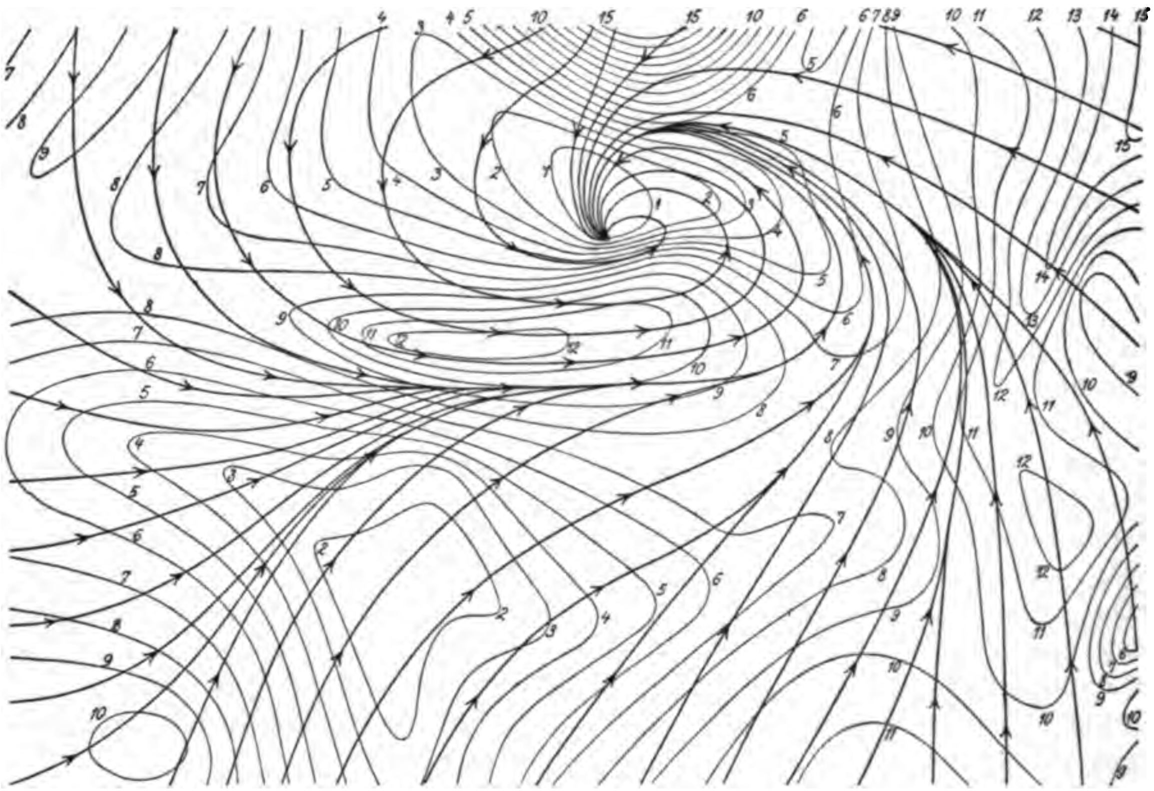


FIG. 102.—Lines of flow and curves of equal wind-intensity, U. S. A., 1905, Nov. 28, 8 a. m.

equal wind-intensity from the preceding chart. We then perform the measurement of the breadth dn of the bands, using the divided sheet of fig. 86. The chart of fig. 104 gives the curves for equal values of these breadths, together with the lines of flow copied from the preceding chart. The graphical multiplication of the field of dn by that of v finally gives the field of transport T_1, which we have represented on the chart of fig. 105 by the following curves

$$T_1 = \ldots 6, 5, 4, 3, \ldots$$

The chart which we have obtained in this manner will represent the horizontal transport in a sheet the thickness of which is given by unit decrease of pressure from the ground to the upper limiting surface of the sheet, and at the same time the vertical transport through this upper limiting surface.

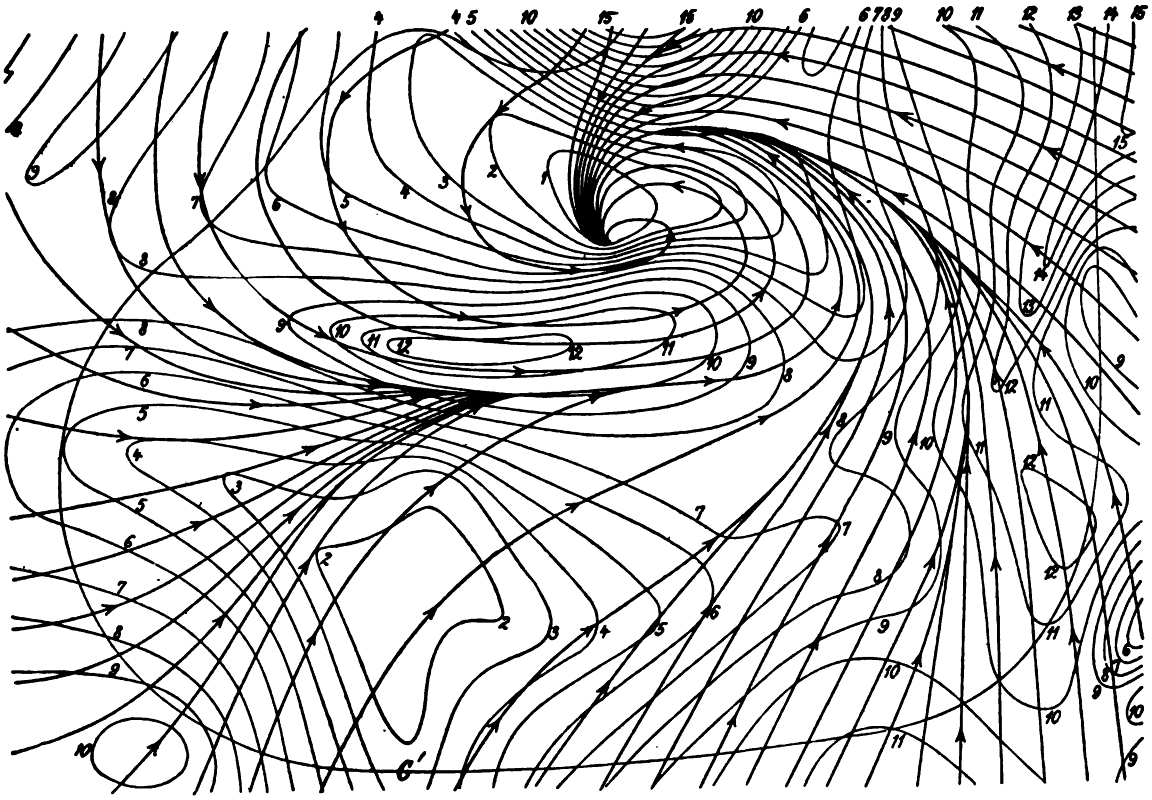


FIG. 103.—Bands of flow which have equal transport at the initial curve C'. U. S. A., 1905, Nov. 28, 8 a. m.

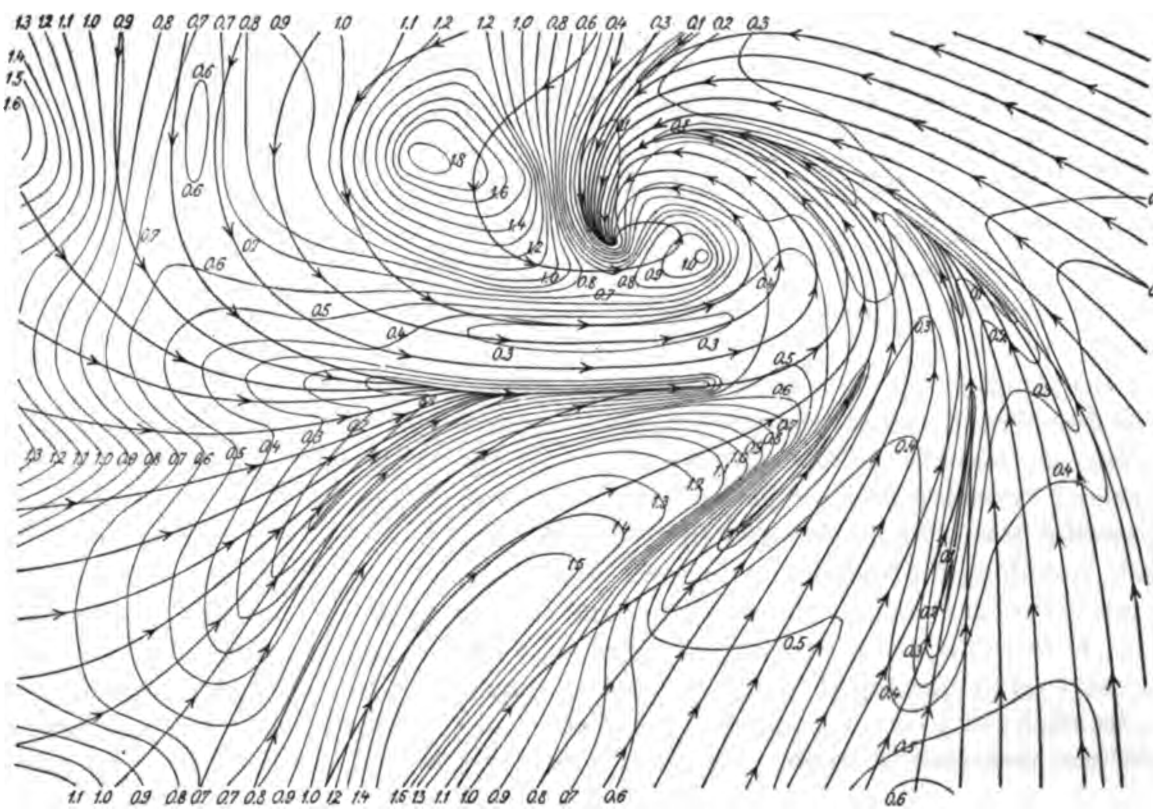


FIG. 104.—Curves for equal breadth dn of the bands of flow. U. S. A., 1905, Nov. 28, 8 a. m.

When we use the decibar as unit pressure the upper limiting surface of the sheet will be situated at the approximate height of 750 meters above the ground. The lines of flow represent vertical walls which divide this sheet into tubes. At the initial wall C' the transport in each tube is 5.10^5 m.t.s. units, *i. e.*, 500,000 tons of air per second. As we proceed from the curve C' to other curves $T_1 =$ const., we have a loss or gain of horizontal transport of 100,000 tons per second. The areas into which the bands of flow are divided by the curves of equal transport will thus represent a vertical transport of 100,000 tons per second through the upper limiting surface of the sheet. This transport is directed upward or downward according

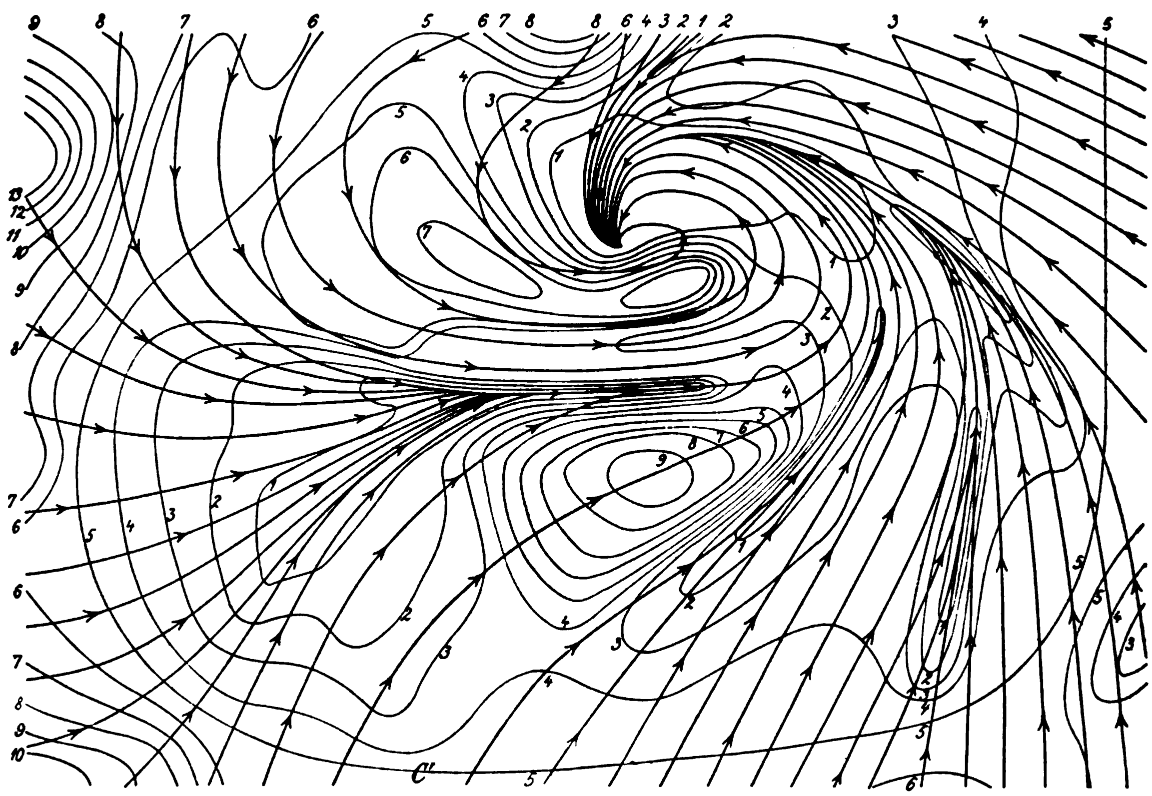


FIG. 105.—Areas of equal vertical mass-transport through a surface where pressure is one unit smaller than at the ground. U. S. A., 1905, Nov. 28, 8 a. m.

as the numbers on the curves $T_1 =$ const. decrease or increase as we proceed in the direction of motion along the tubes. The triangular areas which surround the point and the lines of convergence represent the same vertical transport as the others. As small areas indicate intense vertical motion, we see that we have a powerful ascending motion near the point of convergence, especially on its northern side and along the lines of convergence. But areas of descending motion also occur even very near the point of convergence and between two of the lines of convergence.

If we multiply the pressure of 1 decibar, which defines the sheet, by 0.1 we get a sheet which has the thickness of about 75 meters. The tubes of flow will have a transport of 50,000 tons per second at the wall C', and the areas will represent a vertical transport of 10,000 tons per second through a surface having the approxi-

mate height of 75 meters above the ground. If we multiply by 0.01 we get a sheet of an approximate thickness of 7.5 meters; the tubes will have a transport of 5000 tons of air per second at the wall C', and the areas will represent a vertical transport of 1000 tons of air per second through the surface which has the approximate height of 7.5 meters above the ground. Of course it will be legitimate to go up to so great heights as 75 or 750 meters only on condition that the original chart, fig. 102, represents the average horizontal motion between the ground and these heights.

A change in the interpretation of the charts, which will be useful for qualitative purposes, can be obtained in this manner: we multiply the unit pressure which defines the thickness of the sheet by $\frac{10^n}{750}$. We shall then obtain a sheet the thickness of which will be approximately 1, 10, 100, 1000, . . . meters, according to the value given to n. In order to get the mass-transport in this sheet, we must multiply the field of T_1 by the same number. But instead of that we can multiply only by 10^n on condition of interpreting T_1 as *volume-transport* instead of mass-transport. For 750 is the approximate volume in cubic meters of a ton of air in the lower strata of the atmosphere. In other words, for qualitative purposes it will be permissible to give an interpretation like the following of the chart of fig. 105. It represents a sheet of a thickness of 1000 meters. The tubes have a horizontal transport of 500,000,000 cubic meters of air per second at the wall C', and the areas represent a vertical transport of 100,000,000 cubic meters of air through the surface of a height of 1000 meters. When we choose the thickness of 100 or 10 meters of the sheet, we get the proportional reduction of the numbers representing the volume-transport.

From the chart of fig. 105 we can see without difficulty how the tubes of flow go up and down. Let us return to the original interpretation. The areas of 100,000 tons of vertical transport can then be conceived as the sections of the upper limiting surface of the sheet with tubes of this transport. For each element of the curve C' five such tubes rest upon each other, giving the total horizontal transport of 500,000 tons. Each area shows one of these tubes coming up or going down through the upper limiting surface of the sheet. (Compare the schematic examples of figs. 43 c and 45 c.)

(B) *Topographic method.*—In order to follow not only qualitatively, but quantitatively, the course of the tubes up and down, we can pass to the topographic method. We then retain the curve C', its division into elements fulfilling the condition $v'dn' = 5$ and the corresponding division of the chart into bands of flow, fig. 103. Introducing the value $c' = 5$ in formula (c) section 187, we get

$$-dp_1 = \frac{5}{vdn}$$

By use of the divided sheet for reciprocal length-measurements (fig. 81) we draw the field $\frac{5}{dn}$. The curves representing this field will have the same course as those repre-

senting the breadth dn (fig. 104), only with other intervals. Finally we perform the graphical division by v. The field resulting is given by the chart of fig. 106, represent ing in terms of pressure the topography relatively to the ground of a surface of flow formed by those lines of flow which at the initial curve C' have a height above the ground defined by unit decrease of pressure. The contour-lines $-dp_1 =$ const. of this chart have the same course as the curves $T_1 =$ const. of fig. 105, only with changed intervals. The curves 1, 2, 3, 4, 5, . . . show the points where pressure is 1, 2, 3, 4, 5, . . . units smaller than at the ground. According as we use m-bar, c-bar or d-bar as unit of pressure, these curves will represent the approximate

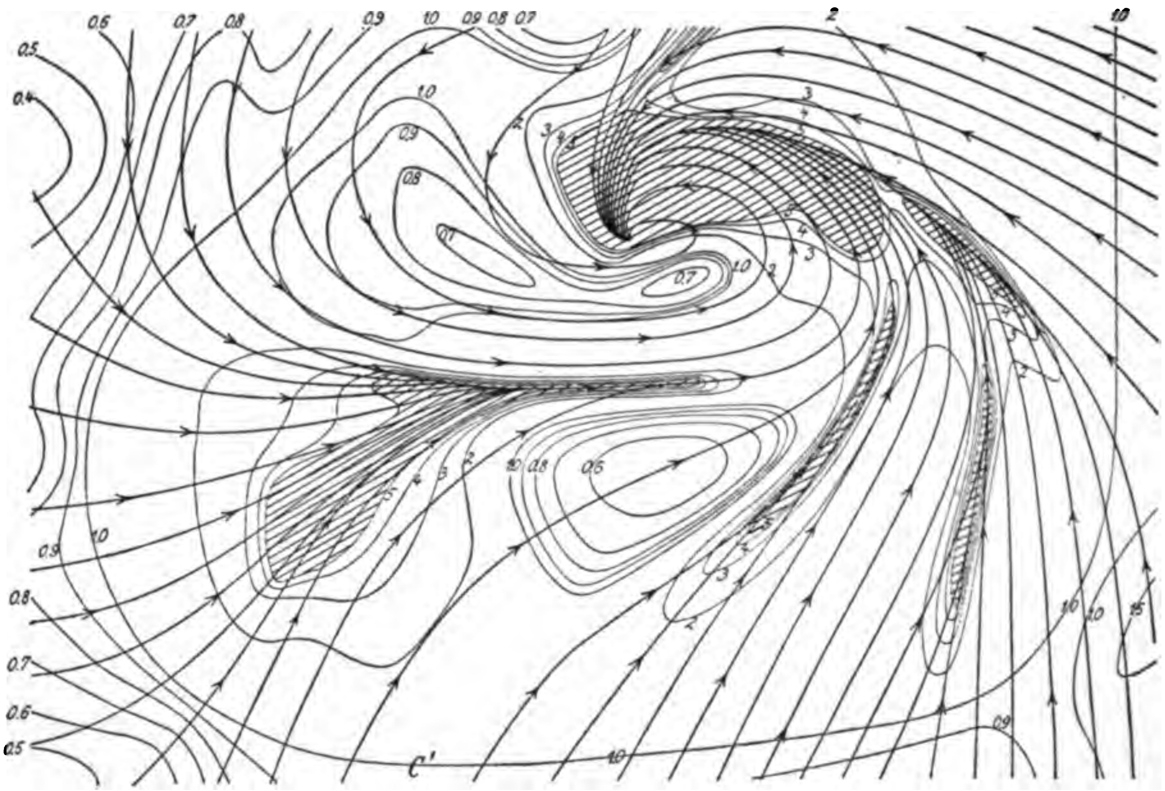


Fig. 106.—Topography of a surface of flow relatively to the earth. U. S. A., 1905, Nov. 28, 8 a. m.

heights of 7.5, 15, 22.5, 30, 37.5, . . . of 75, 150, 225, 300, 375, . . . or of 75, 1500, 2250, 3000, 3750, . . . meters above the ground. Whether it be legitimate to go to greater heights will depend upon whether the given chart gives a true picture of the average horizontal motion between the ground and these heights.

We have drawn no curve inside the curve 5, which, according to the different interpretations, represents an approximate height of 37.5, of 375, or of 3750 meters. But the formal construction, in losing its physical significance, would give an infinity of contour-lines inside this curve, indicating an infinite increase of height of the surface of flow as we approach the point or the lines of convergence. The lowest part of the surface is represented by the curves 0.9, 0.8, 0.7 . . . which are found partly outside the curve C', and partly inside it, especially a little south of the point of convergence and between two of the lines of convergence.

The chart of fig. 106 gives the topography of the surface of flow expressed in terms of pressure; qualitatively we can consider it also as a chart giving topography in terms of height. We have given above the approximate height corresponding to the different integer values of pressure. But if we multiply by $\frac{10^n}{750}$ we pass to decimal heights. Thus in rough approximation we can interpret the curves 1, 2, 3, . . . of the chart as contour-lines which give the heights 1, 2, 3, . . . meters or the heights 10, 20, 30, . . . or 100, 200, 300, . . . of a surface of flow.

(C) *Vertical component of specific momentum.*—In order to find vertical specific momentum, we have to draw a chart of divergence of the horizontal motion (see formula (*h*) of section 187). For this we can use directly the given chart of fig. 102, no special division into bands of flow being required. Divergence of the two-dimensional field of velocity **v** will according to formula (*g*) of section 170 be given by the equation

$$\mathrm{div}_2\, \mathbf{v} = \frac{\partial v}{\partial s} + v\delta$$

s denoting the length of arc along the lines of flow and δ the divergence of these lines (see section 168). As we here come across the most important construction of kinematic diagnosis, we will illustrate each of the four separate operations, the last of which gives the result.

(1) We construct the field of the derivative $\frac{\partial v}{\partial s}$ of the intensity of the vector with respect to its vector-lines. This differentiation is performed in the regular way by use of the differentiating sheet of fig. 81 as illustrated in section 165. The resulting field is given in fig. 107. The numbers added to the curves give the values of the derivative obtained when *ds* is measured in centimeters on the chart. In order to get the true values per meter we have to multiply by 10^{-5}, as a centimeter on the chart represents 10^5 meters.

(2) Then we have to draw the field of divergence δ of the lines of flow. We can determine this field by use of the divided sheet for differentiations of the second order, fig. 90, this sheet being placed with the radii tangential to and the circles normal to the lines of flow. But if the isogons of the lines of flow are given, we get a much better determination by using the ordinary differentiating sheet of fig. 81. We then perform the differentiation of the angle represented by the isogons with respect to the normal curves *n* to the lines of flow. The resulting field of divergence of the lines of flow is given on the chart of fig. 108. The numbers give the value of the divergence referred to the centimeter as unit of length and to the scale of the chart. Multiplying by 10^{-5} we get the true divergence of the lines of flow referred to the meter as unit of length.

(3) Then we perform the graphical multiplication of this field of divergence by that of the intensity *v* of the given velocity. The result of this multiplication, which is performed in the regular way (section 150) is given on the chart of fig. 109.

(4) Finally we perform the graphical addition of the two fields of figures 107 and 109, and change the sign in order to pass from divergence to vertical-component of specific momentum. We thus get the chart of fig. 110, which contains the result.

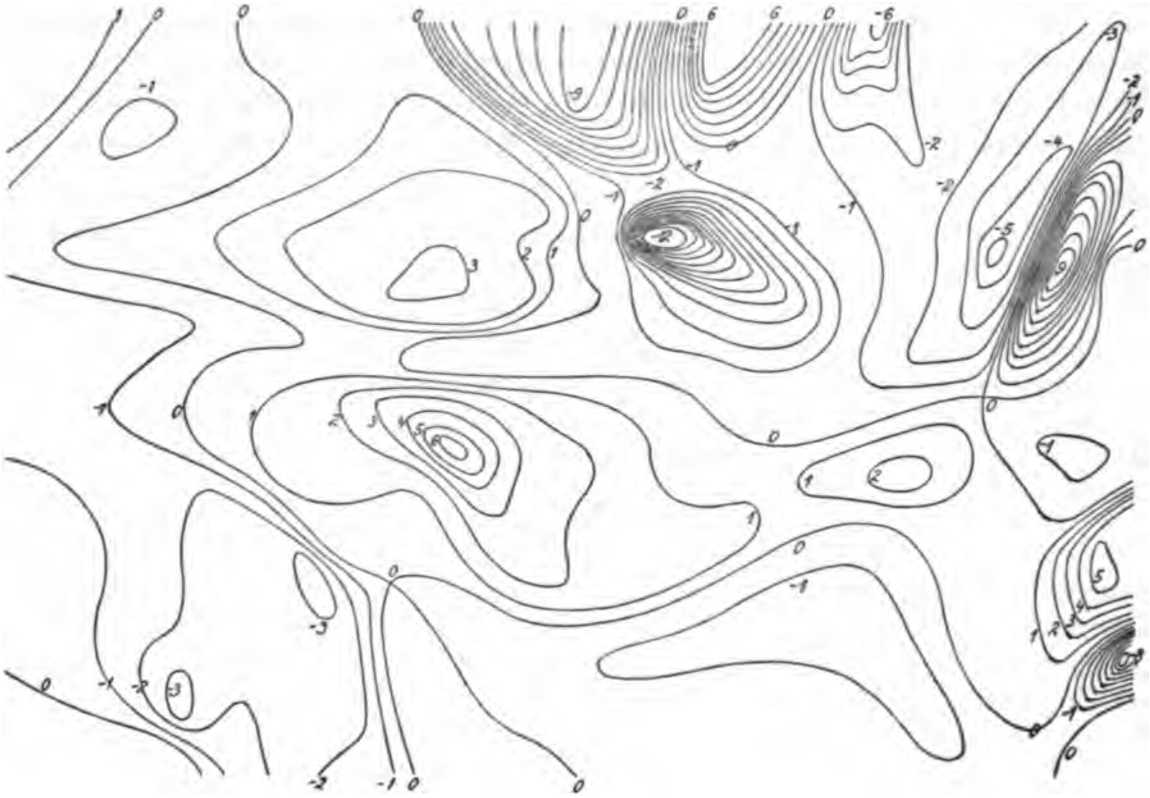


FIG. 107.—Derivative $\frac{\partial v}{\partial s}$ of velocity with respect to the lines of flow. U. S. A., 1905, Nov. 28, 8 a. m.

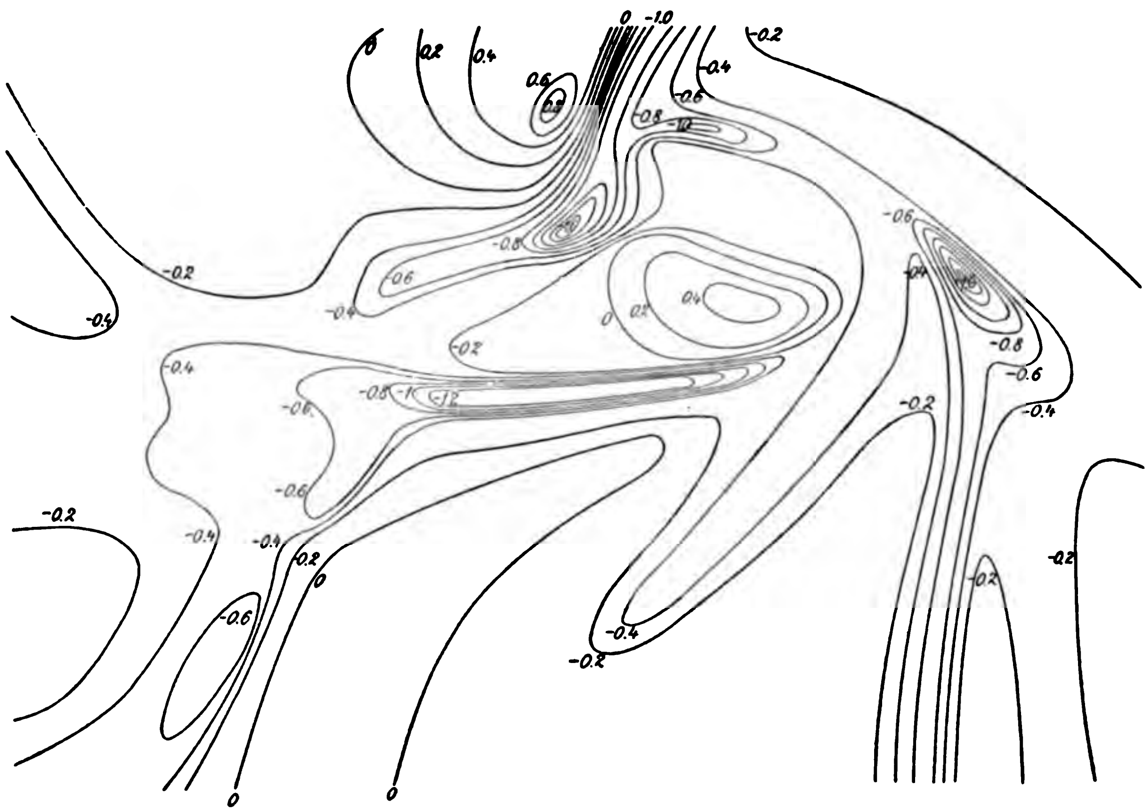


FIG. 108.—Divergence δ of the lines of flow. U. S. A., 1905, Nov. 28, 8 a. m.

FIG. 109.—Product $v\delta$ of wind-velocity and divergence. U. S. A., 1905, Nov. 28, 8 a. m.

FIG. 110.—Vertical specific momentum at a surface, where pressure is one unit smaller than at the ground.
U. S. A., 1905, Nov. 28, 8 a. m.

The chart (fig. 110) gives the vertical component of specific momentum in the height where pressure is one unit smaller than at the ground. The sheet can have a thickness defined by the decrease of pressure of one m-bar, of one c-bar, or of one d-bar. The numbers added to the curves will then represent the vertical specific momentum respectively in the units 0.1 gram per square meter per second, 1 gram per square meter per second, or 10 grams per square meter per second.

Instead of defining the sheets by the decrease of pressure, we can define them as sheets of a thickness of 10, 100, or 1000 meters. The numbers added to the curves on the chart of fig. 110 will then approximately represent *vertical velocity*, in the following units: in tenths of millimeters if the sheet has a thickness of 10 meters, in millimeters if the sheet has a thickness of 100 meters, and in centimeters if the sheet has a thickness of 1000 meters. This rule will be very convenient for getting a qualitative picture of the vertical motion which the chart of fig. 110 describes quantitatively by vertical specific momentum.

The chart is seen to give an ascending velocity which has its greatest values near the point and along the lines of convergence. But areas of descending velocity are also found, even near the point of convergence and between two lines of convergence.

189. Complete Kinematic Diagnosis.—Each of the three methods of representing free vertical motion, by areas of equal vertical transport, by topographic representation of surfaces of flow, or by charts of the vertical component, will have its special advantages in special cases. But the question will now be which of them will work best as a link in a complete kinematic diagnosis of atmospheric motions.

The construction of a chart of areas which represent equal vertical transport will be easy for each single atmospheric sheet. But inasmuch as the lines of flow have a different course in the different sheets, the summation of the transports produced in the different sheets will be circumstantial. For this reason we shall not make a general use of this method.

When the topographic method is applied, we shall not meet with this difficulty. We can pass by simple graphical addition from the relative topographies which we find by the solenoidal condition to the corresponding absolute topographies. But the drawback of the topographic method will be the great complication of the surfaces of flow. In the neighborhood of the initial curve C' used to define the surface it will be relatively simple. But the farther we follow it the more complicated will be the course of the contour-lines. Finally we shall always come to places where the surface folds itself so as to be cut by a vertical line at more than one point. The topographic method of representation will then become complicated, and will lose its conspicuity. While the method may do good service for special investigations, we shall not try to take it as the base for a universal method.

We shall therefore base the complete kinematic diagnosis upon the representation of the vertical motion by charts of the vertical components. The production of these charts is a little more laborious than that of the preceding ones, but as soon as they are produced all further operations will be easy to perform upon them.

In order to perform this diagnosis, we must first know the field of pressure, *i. e.*, we must have charts giving the topography of the standard isobaric surfaces and of the pressure at the ground. From the latter we derive a special chart of the difference of pressure between the ground and the lowest isobaric surface in free air. Then we must have a chart of velocity at the ground, and charts of the average horizontal velocity within each of the standard isobaric sheets, as well as of this average velocity in the incomplete sheet between the ground and the lowest standard surface in free air. The kinematic diagnosis will be accomplished as soon as we have found the complete representation of the vertical motion. We shall arrive at this representation by the following operations (compare the example, section 204, below):

(1) From the chart of velocity in connection with that of pressure at the ground we derive the chart of the forced vertical specific momentum at the ground.

(2) From the chart of the average horizontal velocity in the incomplete sheet between the ground and the lowest standard isobaric surface in free air we derive free vertical specific momentum through this surface. The construction is first performed for a unit sheet, and then the result is obtained for a sheet of irregular thickness by graphical multiplication by the decrease of pressure which defines the sheet.

(3) From the charts of average horizontal velocity in the different standard isobaric sheets we derive the vertical specific momenta produced in each sheet. If a sheet is partly incomplete, the limiting surfaces cutting the ground, we use the method (2) for the incomplete parts of the sheet.

By successive graphical additions of the charts (1), (2), (3), . . . we get the charts of the absolute vertical specific momenta in the different standard isobaric surfaces. If it be desired it will be easy afterwards to change them into charts of vertical velocity.

It will be understood at once how a perfectly similar kinematic diagnosis can be carried out based upon the division of the atmosphere into level instead of into isobaric sheets.

CHAPTER XII.

KINEMATIC PROGNOSIS.

190. Determination of Displacements from Given Velocities.—The fundamental kinematic vector, velocity, is by its very definition a quantity of prognostic nature. If the initial position and simultaneously the velocity of a particle is given, it will always be possible to make a certain definite statement regarding its future position. How far in the future this statement will have any value will depend upon the time-variations of velocity. If it does not vary, either in direction or in intensity, the determination can be made for any future time. But if the velocity varies according to an unknown law, the forecast will be of value only for a limited period of time. When we select a sufficiently short period, the variation of velocity will have insignificant influence, and the prognosis of the future position can be based exclusively upon the knowledge of the initial position of the particle and the initial value of its velocity.

This kinematic prognosis will always be the first step when a rational precalculation of future atmospheric or hydrospheric states is to be made. In principle this step will be perfectly simple. The only delicate point will be the choice of proper periods for which the prognosis may be ventured. They can only be found by experience. As regards the case of the hydrosphere our experience is still quite insufficient. As we have not been able to produce any example of kinematic diagnosis, we can not give any of kinematic prognosis either. As regards the atmosphere, our preliminary experience seems to indicate that periods of a few hours may be used, say from one to six hours. If three hours are used, this period will be convenient also because it is in rough approximation a decimal multiple of our unit of time, the second, viz, 10,800 seconds, or in the mentioned rough approximation 10,000 seconds.

191. Synoptical Representation of Horizontal Displacements.—When a chart of horizontal velocity is given, the tangent to a line of flow gives the direction in which the displacement of any particle takes place, and the scalar value of velocity multiplied by 10,800 gives the length of the displacement in three hours. On the velocity-chart we can thus easily mark the initial and the final situation of any number of points, marking, for instance, the initial position by a little circle with a dark area, and the final position with a corresponding circle with a white area. In order to show which points belong to each other we can draw a line from each black circle to the corresponding white one.

In order to make conspicuous the chart of horizontal displacements, it will be advantageous to choose systematically the initial situations of the points. They can be chosen so that they belong to a set of isogons, or so that they belong to a set of

intensity-curves. In the first case the points situated on the same curve will be displaced in the same direction, in the second along the same length. This will at the same time make the construction easy and the figure conspicuous. A complicated picture will, however, appear in places where one series of points is displaced beyond the initial places of another series.

This difficulty may be completely avoided if we choose the points according to another principle, namely, so that the final situation of one point shall be the initial situation of another. In this manner we get chains of points (fig. 111) which have a certain similarity with the lines of flow and would coincide with them if we drew the displacements for infinitely short intervals of time.

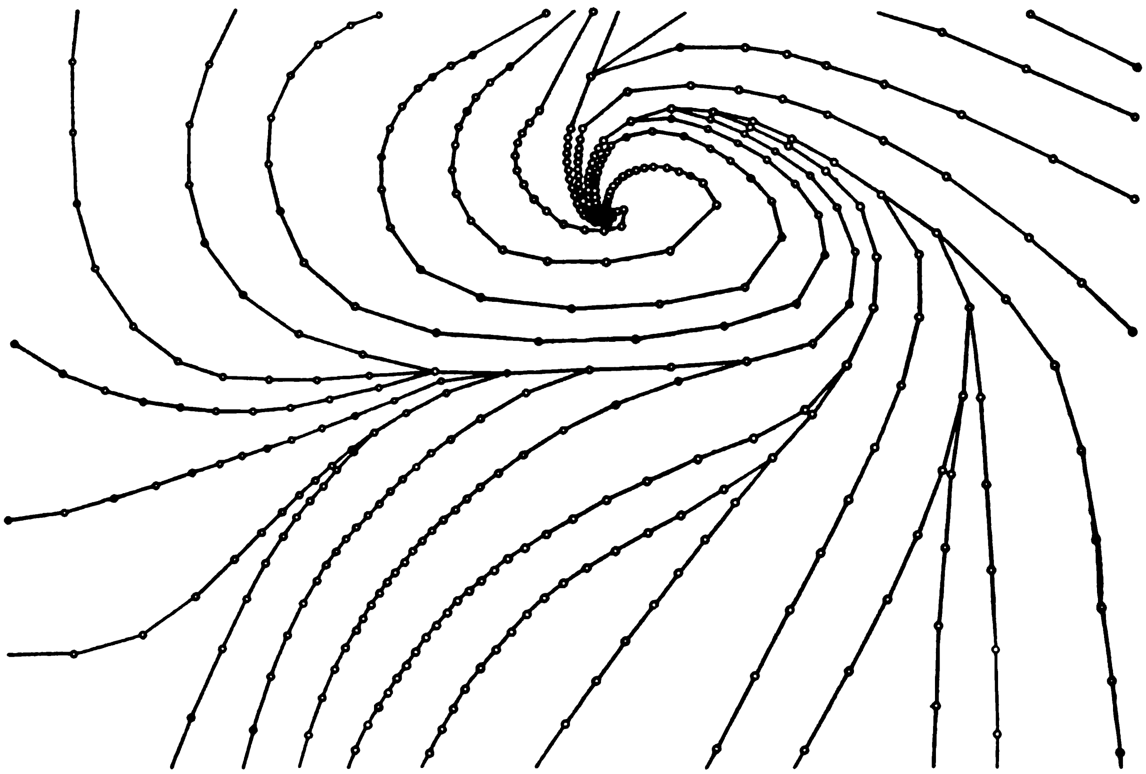


FIG. 111.—Displacements in 3 hours. U. S. A., 1905, Nov. 28, 8 to 11 a. m.

It will be understood at once that from the corresponding charts of vertical velocity we can derive the correlated vertical displacements, but it will be of no use to enter into details before we come to the more general problem of dynamic prognosis. It will be sufficient that we have indicated here the general principle of kinematic prognosis.

192. Different Forms of the Equation of Continuity.—Before we leave the question of kinematic prognosis we have to examine the prognostic value of the equation of continuity. We have already alluded to the prognostic nature of this equation, but we have used it hitherto exclusively for diagnostic purposes. For the more general purpose we have first to give the complete mathematical formulation of the equation of continuity. The two theorems, section 114 (A) and (B),

correspond to two different mathematical forms of the equation. The first theorem deals with the velocity of expansion of a given mass and states its identity with the integral of the normal component of velocity taken over the limiting surface of the mass. K being the volume of the mass, the velocity of expansion will be expressed by the *individual* time-derivative of K. Thus we can write the equation

$$\text{(a)} \qquad \frac{dK}{dt} = \int v_n \, d\sigma$$

The theorem 114 (B) deals with the varying mass M which is contained within a stationary volume, and states that the diminution of this mass per unit time is equal to the mass-outflow through the limiting surface of the volume. Evidently the diminution of the mass M per unit time in a stationary volume is expressed by the negative *local* time-derivative of M. When we identify this derivative with the well-known expression of the mass-outflow, we get this other form of the equation of continuity

$$\text{(b)} \qquad -\frac{\partial M}{\partial t} = \int V_n d\sigma$$

In order to bring the equations to forms more easily used we can apply them to infinitely small volumes K. The integrals appearing in the second member of (*a*) or (*b*) will then be expressed by the product of this volume K into the divergence of the vector. When at the same time we express the volume K of the moving mass by the product of its mass M into its specific volume, $K = \alpha M$, and remember that this mass M is constant, we get equation (*a*) in the form

$$M\frac{d\alpha}{dt} = K \operatorname{div} \mathbf{v}$$

Dividing by K, and remembering that the ratio $\frac{M}{K}$ is the reciprocal specific volume, we get

$$\text{(c)} \qquad \frac{1}{\alpha}\frac{d\alpha}{dt} = \operatorname{div} \mathbf{v}$$

In the same manner, when in (*b*) we express the mass M as the product of its density ρ into its volume K, and remember that here the volume is stationary in space and therefore constant, we get

$$\text{(d)} \qquad -\frac{\partial \rho}{\partial t} = \operatorname{div} \mathbf{V}$$

When we use the relation existing between local and individual derivative (section 177) as well as the relations existing between density and specific volume and between velocity and specific momentum, we can verify at once the fact that (*c*) and (*d*) are merely different forms of the same equation.

193. Equation of Continuity as a Prognostic Equation.—Equation (*d*) of the previous section directly tells us that if we know the field of specific momentum at any moment, we can find a field representing the rate of decrease of density $-\frac{\partial \rho}{\partial t}$

simply by forming the field of divergence of the specific momentum. Then we could multiply the field of $-\frac{\partial \rho}{\partial t}$ by a suitable interval of time dt, and add it to the field of density at the time t. We should then get the field of density at the time $t+dt$.

This method could be formally carried out if we had sufficiently complete and exact observations of specific momentum $\mathbf{V}$. But as we have to form the divergence in space, we need observations not only of the horizontal, but also of the vertical component of specific momentum. Therefore, as long as we can get an idea of the vertical motions only in the indirect way by making a diagnostic use of the equation of continuity, supposing simply the field of density to be stationary in space, $\frac{\partial \rho}{\partial t} = 0$, every prognostic use of the equation of continuity in this direct way will be excluded.

But we could also think of a prognosis of a more summary character, which would also be of great value if it could be carried out practically. We shall return to the equation of continuity in the integral form

$$(a) \qquad -\frac{\partial M}{\partial t} = \int V_n d\sigma$$

and apply it to a vertical cylinder going from the ground to the limit of the atmosphere, or at least to a height in which the density of the air is so low that it can cause only an insignificant mass-transport. It will then be sufficient to integrate the horizontal specific momentum over the cylindrical surface, and our ignorance of the vertical motion will cause no difficulty.

Now the ground carries the weight of the mass of air M in this cylinder. σ_0 being the area of the base and p the pressure, we have $p\sigma_0 = Mg$, g representing an average value of acceleration of gravity. Multiplying equation (a) by g, introducing $p\sigma_0$ instead of gM, and remembering that the cylinder is stationary and therefore its base σ_0 constant, we see that the equation can be written

$$(b) \qquad -\frac{\partial p}{\partial t} = \frac{g}{\sigma_0}\int V_n d\sigma$$

Therefore, if we know horizontal specific momentum $\mathbf{V}$ sufficiently well up to sufficient heights, we should be able by this equation to forecast the change of pressure at the ground. Evidently this would be of high practical value.

The question whether this will succeed will depend on the degree of completeness and of accuracy required in the knowledge of $\mathbf{V}$, or of the corresponding velocity $\mathbf{v}$. In order to estimate it, we can express the vertical dimension by pressure and at the same time substitute velocity for specific momentum. Thus we have first $d\sigma = dz\, ds$, dz being a vertical and ds a horizontal element of line. Then we can express dz approximately by pressure, writing $dz = -0.1\ \alpha\ dp$, where z is measured in meters and p in the m. t. s. unit of pressure, centibar. Thus

$$\int V_n d\sigma = \int\int V_n dz ds = \int\int v_n(-0.1\ dp)\ ds$$

Here we can first perform the integration with respect to p, denoting by $\bar{v}_n$ the average value of v_n along a vertical line of the cylindrical surface, and by $p-p'$ the difference of pressure between bottom and top of the cylinder. Then

$$\int v_n d\sigma = 0.1\,(p-p')\int \bar{v}_n ds$$

Finally we can by $\bar{\bar{v}}_n$ denote the average value of $\bar{v}_n$ when we integrate with respect to s, *i. e.*, around the base of the cylinder. The circumference being s, we get

$$\int v_n d\sigma = 0.1\,(p-p')\, s\bar{\bar{v}}_n$$

Equation (b) thus takes the form

$$-\frac{1}{p-p'}\frac{\partial p}{\partial t} = 0.1\frac{s}{\sigma_0}g\bar{\bar{v}}_n$$

If the cylinder is circular, the ratio of its circumference to the area σ_0 of its base will be $\frac{4}{D}$, D being the diameter of the base. As p' is the pressure in very great height, and thus very small, we can leave it out without essentially changing the formula. Thus we get for a circular cylinder of sufficient height

$$-\frac{1}{p}\frac{\partial p}{\partial t} = 0.4\frac{g}{D}\bar{\bar{v}}_n$$

In order to estimate the exactitude required in the observations of velocity if it should be possible to forecast pressure at the ground by this formula, we solve with respect to $\bar{\bar{v}}_n$

$$\bar{\bar{v}}_n = -\frac{1}{p}\frac{\partial p}{\partial t}\frac{D}{0.4\,g}$$

In this formula we can express pressure in any unit. We shall then use m-bars. Passing at the same time from second to hour as unit of time, calling m the change of pressure in m-bars per hour, setting $g = 9.8$, and calling d the diameter of the cylinder expressed in kilometers, $d = 0.001\,D$, we get

$$\bar{\bar{v}}_n = -0.00007\,md$$

For the change of pressure of 1 m-bar per hour, $m = 1$, we shall then have

$d = 1000$ km.	$\bar{\bar{v}}_n = 7$ cm. per second.
$d = 100$ km.	$\bar{\bar{v}}_n = 7$ mm. per second.
$d = 10$ km.	$\bar{\bar{v}}_n = 0.7$ mm. per second.

Thus, even if we take areas of a diameter of 1000 km., the observations of the wind-velocity would have to be correct to a centimeter over the whole area of a cylinder having this diameter and extending up to heights where pressure is imperceptible. Of course observations of wind-velocity of this exactitude and completeness can not be thought of in the present state of development of meteorological observations.

Kinematic prognosis could therefore hitherto only give the displacements of the masses of air as developed in sections 190 and 191.

CHAPTER XIII.

REVERSAL OF THE PROBLEM OF KINEMATIC PROGNOSIS. KINEMATIC DETERMINATION OF ACCELERATION.

194. On the Reversed Problems.—According to our general plan (section 96) we shall now consider the problem of kinematic prognosis in its reverse form. Knowing from the observations the initial and the final state of motion, we shall investigate the change of motion which has led from one state to the other. This will involve a determination, on pure kinematic principles, of the acceleration of the moving particles.

If we ever succeed in giving the complete solution of the problem of prognosis, we shall have to determine the accelerations not by kinematic but by dynamic methods. This should be theoretically possible because the observations should allow us to derive the forces which produce the accelerations. But passing to the practical performance, we shall meet with a great difficulty. Though we know from laboratory experiments the coefficient of the friction of the air, we shall not be able to use it practically for determining the influence of friction on acceleration. The reason is obvious. Friction depends upon true motion, while we are forced to work with an idealized motion, disregarding all the small irregularities of the motion rsection 97) and of the ground (section 179). If we were to determine the frictional resistance in the rational way we should have to examine the motion from millimeter to millimeter, and not only at stations which may be hundreds of kilometers from each other. As this will not be possible, we shall be obliged to find other ways for determining the influence which, as an indirect effect of friction, modifies the idealized motion which we consider. We must develop methods for determining, by pure kinematic principles, the acceleration of the idealized motion to the consideration of which we are confined, and by comparing it with the accelerating forces find empirical rules for taking the effect of friction into account.

As an introductory problem to the kinematic determination of accelerations, we shall first treat the problem of the identification of particles on two successive charts of motion or (what comes to the same thing) the determination in the second approximation of the displacement of these particles.

195. Determination of Displacements in the Second Approximation.—Let a chart be given which represents the state of motion at the epoch t_0. We shall consider a particle which at this epoch is situated at the point A (fig. 112). According to the chart it has a certain velocity $\mathbf{v}_0$. During the short interval of time t_1-t_0 its displacement will then in the first approximation be

$$(a) \qquad AB' = \mathbf{v}_0\,(t_1-t_0)$$

Thus the point B' will give in the first approximation the situation of the particle at the epoch t_1.

Now let a second chart be given, which represents the state of motion at this second epoch t_1. According to this chart we shall find not the velocity $\mathbf{v}_0$ but a certain velocity $\mathbf{v}_1'$ at the point B'. We may therefore expect to get a better determination of the displacement if we suppose that the particle has moved not with the velocity $\mathbf{v}_0$, but with the velocity $\frac{1}{2}(\mathbf{v}_1'+\mathbf{v}_0)$. During the time t_1-t_0 this velocity would give the displacement

(b) $$AB'' = \tfrac{1}{2}(\mathbf{v}_0+\mathbf{v}_1')(t_1-t_0)$$

The point B'' would then more correctly than the point B' give the situation of of the particle at the epoch t_1.

The point B'' and the corresponding displacement AB'' can be found by a direct continuation of the construction which led to the point B'; *i. e.*, we measure off from this point the displacement

(b′) $$B'C'' = \mathbf{v}_1'(t_1-t_0)$$

The point B'' will then be the central point of the line AC'' which represents the vector-sum of the two displacements AB' and $B'C''$.

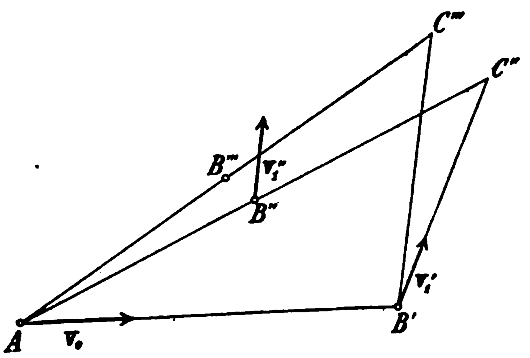


FIG. 112.—Construction of displacement in second approximation.

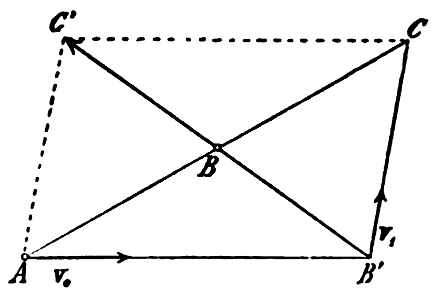


FIG. 113.—Displacement and acceleration.

But according to the second chart the velocity at the point B'' will not have the value $\mathbf{v}_1'$ but a certain other value $\mathbf{v}_1''$. Therefore we may expect to get a still better determination of the displacement required if we suppose that the particle has moved during the time t_1-t_0, not with the velocity $\frac{1}{2}(\mathbf{v}_0+\mathbf{v}_1')$, but with the velocity $\frac{1}{2}(\mathbf{v}_0+\mathbf{v}_1'')$, which would have given the displacement

(c) $$AB''' = \tfrac{1}{2}(\mathbf{v}_0+\mathbf{v}_1'')(t_1-t_0)$$

The point B''' at which the particle would then arrive can be found by a construction similar to that which led to the point B''. From the point B' we set off the displacement

(c′) $$B'C''' = \mathbf{v}_1''(t_1-t_0)$$

The point B''' will then be the central point of the line AC''' which represents the vector-sum of the two displacements AB' and $B'C'''$.

Evidently these constructions can be continued indefinitely. Two cases may then present themselves:

(1°) The distances between the points B', B'', B''' . . . may converge toward zero. The process of constructions will then be convergent and will lead ultimately

to the determination of a definite point B. This point will then represent the situation of the particle at the epoch t_1 with the highest degree of approximation which can be attained when the determination is to be made by the use of *two* charts of velocity instead of by the use of only one. Or in other words: AB will represent the displacement in the second approximation.

(2°) The distances between the points in the series B', B'', B''' . . . may remain finite. The process of constructions will then be divergent and will lose every physical significance. Examples of this divergence can easily be given. We should meet with it, for instance, in the case of atmospheric wave-motions (see fig. 51) if the interval of time $t_1 - t_0$ was selected of such a length that the displacement (a) obtained in the first approximation was of the same order of magnitude as the wave-length. This case of divergence must be avoided, and can always be avoided if the selected interval of time $t_1 - t_0$ be sufficiently short. The periods which from this point of view may be used must be found gradually by experience.

We shall consider henceforth exclusively the case of convergence, and of convergence so rapid that already the point B'' or the point B''', will define with sufficient approximation the situation of the required point B. According to our experience the interval of three hours which we have used seems always to give convergence, and as a rule of a satisfactory rapidity.

196. Discontinuous Method for Constructing Charts of Acceleration.—Let A be the situation of the considered particle at the epoch t_0, and B its situation at the time t_1 as we find it in the second approximation by the construction of the preceding section. $\mathbf{v}_0$ being the velocity at the point A at the epoch t_0, and $\mathbf{v}_1$ the velocity at the point B at the epoch t_1, we shall then have (fig. 113)

$$(a) \qquad AB' = \mathbf{v}_0(t_1 - t_0) \qquad B'C = \mathbf{v}_1(t_1 - t_0)$$

the half vector-sum of these displacements AB' and $B'C$ defines the displacement AB.

The vector-difference of the same two displacements (a) will be represented by the line $B'C'$, for which we shall thus have

$$(b) \qquad B'C' = (\mathbf{v}_1 - \mathbf{v}_0)\,(t_1 - t_0)$$

Now let us divide this equation by $(t_1 - t_0)^2$. We shall then get

$$(c) \qquad \frac{\mathbf{v}_1 - \mathbf{v}_0}{t_1 - t_0} = \frac{B'C'}{(t_1 - t_0)^2}$$

But the first member in this equation represents the average acceleration of the particle during the time $t_1 - t_0$. The equation therefore expresses the fact that the vector $B'C'$ which we have constructed will, after division by the square of the interval of time $(t_1 - t_0)^2$, represent the acceleration required. If this acceleration should be attributed to a definite place in the field, it would of course be to the central point between the points A and B.

We have thus arrived at a discontinuous method of constructing charts of acceleration: For a sufficient number of particles we perform the construction giving the displacement of the particles in the second approximation. This construction at the same time gives the vector $B'C'$, which gives the direction and

(after division by the square of the interval of time) the intensity of the acceleration. When we have constructed this vector at a sufficient number of points, we can afterwards draw its vector-lines or its isogons and its intensity- curves.

197. Continuous Method for Constructing Charts of Acceleration.—We can base a continuous method of constructing accelerations upon the analytical representation of this vector as a complex time-derivative and space-derivative. By formula *f* of section 177 we have

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\nabla\mathbf{v}$$

We have already called the first term of the second member the *local* acceleration. If this term be zero

$$\frac{\partial \mathbf{v}}{\partial t} = 0$$

the wind-fanes of each station will show invariable direction and the anemometers invariable intensity of the wind. The velocity-chart will remain unchanged as long as this condition is fulfilled. The particles of air will then move along a system of lines of flow which remain unchanged. The lines of flow will be the paths of the moving particles. During this motion the particles will accelerate or retard so as to take at every place precisely the velocity which is characteristic of the place. We shall call a motion which is defined by this condition a *stationary* motion. The acceleration which the particles of air must have in the case of stationary motion is obtained if we set the local acceleration equal to zero in equation (*f*) of section 177; *i. e.*, the term $\mathbf{v}\nabla\mathbf{v}$ represents the acceleration which the particles must have if the motion is stationary.

We can therefore state: The acceleration of the moving particles may be represented as the vector-sum of two partial accelerations:

(A) Stationary acceleration which is given by the space-derivative

$$\mathbf{v}\nabla\mathbf{v} \tag{a}$$

(B) Local acceleration which is given by the time-derivative

$$\frac{\partial \mathbf{v}}{\partial t} \tag{b}$$

We have already treated the construction of fields representing derivatives of the forms (*a*) and (*b*). We can thus construct the fields of the two partial accelerations, and form their vector-sum

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\nabla\mathbf{v} \tag{c}$$

which will then give the field of the true acceleration.

(A) *Stationary acceleration.*—When a velocity-field is given, the field of stationary acceleration (*a*) can be found in the following way (see section 174).

(1) We perform the derivation of the half square of velocity with respect to the lines of flow, *i. e.*, we form the field

$$\frac{\partial}{\partial s}\left(\tfrac{1}{2}v^2\right) \tag{d}$$

It will easily be seen that this field represents the tangential component of the acceleration in the stationary motion.

(2) We form the field of curvature γ of the lines of flow, and perform the multiplication of this field by that of the square of the velocity v^2. The field

$$\gamma v^2 \tag{e}$$

which we get in this way evidently represents the normal component of acceleration in the stationary motion of the particles along the lines of flow.

(3) We perform the vector-addition of the vector (d) which is directed along, and the vector (e) which is directed normally to the lines of flow.

Another method of constructing the field of stationary acceleration in which the single operations will not have the same simple physical significance, but which may still under special circumstances be advantageous, will be this (see formula (d') of section 174):

(1′) We construct the ascendant of the half square of the velocity

$$\nabla(\tfrac{1}{2}v^2) \tag{d'}$$

(2′) We construct the two-dimensional curl of the velocity (section 172) and form the vector-product of this vector and the velocity. This vector

$$(\mathrm{curl}_2\mathbf{v})\times\mathbf{v} \tag{e'}$$

will be directed along the positive normal to the lines of flow.

(3′) We perform the vector-addition of the two vectors (d') and (e').

(B) *Local acceleration.*—While stationary acceleration is derived from *one* chart which represents the given field of velocities at the given time, local acceleration must be derived from *two* charts which represent velocity at the two different epochs. The method will be that of the regular vector-subtraction and subsequent division by the interval of time as we have developed for the case of pure time-derivations of vector-fields (section 176).

198. Special Remarks.—The chart of local acceleration which we derive from the charts of velocity for the epochs t_0 and t_1 will correspond to the epoch $t_0+\frac{1}{2}(t_1-t_0)$. On the other hand the chart of stationary acceleration, which is derived from one of the given charts of velocity, will correspond either to the epoch t_0 or to the epoch t_1. If the interval between these epochs is sufficiently short, the circumstance that the charts of local and of stationary acceleration correspond to slightly different epochs will cause no trouble. But in order to get a satisfactory construction of the local acceleration, we are obliged to select the interval of time t_1-t_0 with as great a length as possible. For this reason it will be rational to derive the stationary acceleration from both given charts of velocity. The best method will then be this:

By vector-addition and division by 2 we form the chart of the average velocity

$$\tfrac{1}{2}(\mathbf{v}_1+\mathbf{v}_0)$$

during the time from t_0 to t_1. From this chart of average velocity we derive the chart of stationary acceleration, which will then correspond to the epoch $t_0+\frac{1}{2}(t_1-t_0)$. Then we form the vector-difference of the same two fields of velocity

$$\mathbf{v}_1-\mathbf{v}_0$$

divide by the interval of time $t_1 - t_0$, and thus find the local acceleration at the epoch $t_0 + \frac{1}{2}(t_1 - t_0)$. Then the sum of the two partial accelerations will give the best value of the acceleration at this epoch.

As even the determination of stationary acceleration involves a vector-addition, the complete determination of the field of acceleration will involve the performance of no less than two vector-additions and one vector-subtraction. The work will therefore continue laborious. But as this kinematic determination of accelerations will never enter as a link in the chain of operations which must be performed for the solution of the problem of prognosis, a practical demand for special rapid methods will not be required.

199. Return to the Problem of Prognosis.—It may be useful to consider a little more closely what could be done for the problem of prognosis as soon as we can determine by dynamic methods the accelerations of the moving particles.

To the displacement AB' (fig. 113) found in the first approximation we should then be able to add the displacement $B'B$ due to the acceleration. In this manner we should be able to forecast the displacements of the particles with a higher degree of approximation, retaining the length of time for which we make the forecast; and, dispensing with the greater accuracy, we could make the forecasts for longer periods.

But in addition to this we should also be able to prognosticate the new field of motion. For we know the velocities which the particles have when they arrive at their new positions, and we can then draw the field of these velocities. Instead of this discontinuous method we could also use a continuous one. From the field of velocities observed we should have to derive that of stationary acceleration. Subtracting this field from that of the true accelerations, which we calculated by dynamic methods, we will get the field of local acceleration. Multiplying this by a suitable interval of time $t_1 - t_0$, and adding to the field of velocity at the time t_0, we get the field of velocity at the time t_1. Thus, as soon as the dynamic method has given us the field of accelerations, kinematic methods, which we have treated already, allow us to determine the future field of horizontal motion. From this we may again, by kinematic methods which we have developed, derive the correlated vertical motions.

CHAPTER XIV.

EXAMPLES OF ATMOSPHERIC MOTIONS.

200. Indian Southwest Monsoon in July.—In giving a few examples of the kinematic diagnosis of atmospheric motions, we shall begin with a case of great simplicity, namely, the Indian Southwest Monsoon in July, at the time of its highest development.

Plate XXXI gives the discontinuous representation of this air-motion taken from plate 17 of Sir John Eliot's Climatological Atlas of India. The arrows represent the average wind-directions for the month, and the numbers the corresponding average intensities, changed from miles per hour to meters per second. In this case the distribution of arrows is a regular one, and it causes no difficulty to draw the lines of flow from them. The moderately idealized contour-lines of the blank map on which the construction is performed are a good help for the understanding and correct drawing of these lines. The chart representing the lines of flow and curves of equal wind-intensity is given on plate XXXII.

On the peninsula of India the motion represented by these two systems of curves is of great regularity. A striking effect of the topography of the land is seen, inasmuch as the lines of flow make a bend around the southern projection of the peninsula in order to avoid going across the mountains of the west coast. In the places where the wind must still travel directly toward the shore and the slope of the mountains, decided minima of velocity are seen to exist.

In the northern part of the chart the most marked peculiarity is the long line of convergence which goes up the whole length of the Ganges valley, in order to end in a constellation of a point of convergence and a neutral point situated above the Punjab plains. This long line of convergence is evidently an effect of the Himalaya chain. The observations do not go to a sufficient height in the mountains to let us see the complete character of the motion. But in all probability a correlated line of divergence must exist higher up on the slope of the chain. These two parallel lines of convergence and of divergence will then give the limits in horizontal projection of a rolling mass of air, which is kept in rotatory motion by the Monsoon-current passing across the mountain in greater height (cf. fig. 52 B, p. 59).*

Plate XXXIII shows the forced vertical velocity at the ground. This chart has been derived from the preceding one by the method described in section 181. The shaded parts are the areas of ascending and the unshaded ones those of descending motion, the shaded ones on the windward and the unshaded ones on the leeward sides of the mountains. The ascending motion reaches its greatest values on the west coast, where it has a maximum amounting to 15 cm. per second.

Plate XXXIV gives the free vertical motion derived by use of the solenoidal condition as described in section 188 (C). Properly adjusting the units, we can

*Mr. E. Gold has arrived at a similar conclusion. Nature, Feb. 1908, p. 355.

interpret the chart as representing vertical specific momentum in the height where the pressure is one unit smaller than at the ground, or vertical velocity in unit height above the ground. If we use the latter interpretation, the numbers added to the curves give the vertical velocity in millimeters per second at the height of 100 meters above the ground, and in centimeters per second at the height of 1000 meters above the ground. The chart will give the correct picture of this part of the vertical velocity, provided that the chart of plate XXXII represents the average horizontal motion for the sheet between the ground and these heights. For a wind which has the regularity of the monsoon, it is not improbable that the observations at the ground give the character of the motion up to considerable heights. But decided exceptions exist. Thus, if the line of convergence in the Ganges valley existed unchanged to the height of 1000 meters it should give here a vertical velocity of 9 cm. per second, and a corresponding greatly localized precipitation might be expected. But as Sir John Eliot's chart of precipitation for July does not show any sign of this, we have a strong reason for believing that the line of convergence is a local phenomenon limited to the lower layers. (Compare section 134.)

Comparing the plates XXXIII and XXXIV, we see that the free vertical motion has a certain tendency to be of an opposite sign to the forced vertical motion existing at the ground. The addition will therefore in most places give a reduced vertical motion. Lower down the forced vertical velocity is the stronger of the two. But at the height of 1000 meters both are of about the same order of magnitude and as we proceed farther upward the influence of the ground will constantly recede to the background.

From the two charts XXXIII and XXXIV we can derive charts for the total vertical velocity at any constant height above the ground by graphical addition. If we wish to have the total vertical velocity at a given height above sea-level we must, before the addition, perform the graphical multiplication of chart XXXIV by a chart which represents the height from the ground to the given level. It is interesting to draw such charts of total vertical motion and to compare them with charts of average precipitation like those found in Eliot's Atlas. But in a case like that before us no complete accordance should be expected. We have referred already to one departure, the reason of which is easily understood. Another cause of departures is this: In spite of its great regularity the monsoon-wind shows changes from day to day, causing corresponding changes from day to day in the distribution of the vertical motion. For this reason there will from time to time appear ascending motion and consequently precipitation in places where the average motion is descending and where no precipitation would appear if there were no departures from the average motion.

201. North America, 1905, November 28, 8 a. m.—Instead of average motions we shall henceforth consider actual motions.

Plate XXXV represents the field of pressure and of mass in the lowest atmospheric sheet above North America, November 28, 1905, 8 a.m., time of 75th meridian. The single lines give the absolute topography of the 1000 m-bar surface, and the

double lines, consisting of a thick line and a thin one, give the relative topography of the 900 m-bar surface relatively to the 1000 m-bar surface, and thus the average specific volume of the air in the sheet between these two surfaces. The thin line is on the side where the sheet is thinner. All lines are stippled where they have their course below the ground. It will be seen that the 1000 m-bar surface has a strong depression, going down to 100 dynamic meters below sea-level in southern Minnesota, with a secondary depression in Colorado. The great area of depression is surrounded by high areas situated in New England, in Montana and the adjacent parts of Canada, and on the southern part of the coast of California. Another depression is situated farther north on the Pacific coast.

Plate XXXVI gives the representation of the observed wind-directions in the common way by arrows. The corresponding numbers, according to the dial of fig. 32, are also inscribed, and another set of numbers give the wind-intensities in meters per second. A glance at the arrows at once shows the unfortunate consequences of the observation of only eight wind-directions. If the lines of flow were drawn strictly tangential to the arrows they would get polygonal form, with a great number of lines of convergence and of divergence separating from each other the areas of different wind-directions. It must therefore be highly recommended to observe at least double the number of wind-directions. Provisionally we can only remove the discontinuities in the drawing of the isogonal curves or the lines of flow by eye-measure.

Plate XXXVII gives the continuous representation of the motion by isogonal curves and curves of equal wind-intensity. Plate XXXVIII gives the same representation by lines of flow and intensity-curves. The isogonal curves have a remarkably simple course: only two singular points appear, one in southern Minnesota and one in California—the former positive, the latter negative. The lines of flow show a marked point of convergence in southern Minnesota, near the point of the lowest depression, and several lines of convergence which run into this point. A line of divergence connects the two high areas in Montana and California, and this line has a neutral point where the isogonal curves had the negative singular point. The lines of flow make a very striking bend in order to go around instead of across the Allegheny mountains. While the lines of flow have a relatively simple course, the distribution of wind-intensity is very irregular, with a great number of maxima and minima. North of the cyclonic center winds go up to 28 meters per second.

Chart XXXVIII is drawn upon a blank surface which gives the topography of the land greatly idealized. By the method of section 181 we have derived from it the chart of plate XXXIX, which gives the vertical velocity at the ground. The shaded areas on the windward slopes are those of ascending motion, the unshaded ones on the leeward slopes those of descending motion. The greatest vertical velocities amount to 20 cm. per second. The fact that higher values are never reached is of course due to the idealization of topography. The true local values may be much greater, while our chart gives only the average values for greater areas.

From either of the two plates XXXVII and XXXVIII we can derive by the solenoidal condition the free vertical motion, which is represented by the chart of

plate XL. We can interpret the chart as giving vertical velocity in millimeters per second at the height of 100 meters above the ground, or in centimeters per second at the height of 1000 meters above the ground. If we venture to extrapolate to the latter height we get free vertical velocities of the same order of magnitude as the forced vertical velocity derived by the surface-condition. This free vertical velocity is seen to have a very irregular distribution. A certain tendency to be opposite to the forced one is manifest in different places. But as to its general features free vertical motion is seen to be governed by pressure. Generally speaking the area of depression is an area of ascending motion, except in the details, inasmuch as smaller areas of descending motion exist even in the immediate neighborhood of the cyclonic center.

We can in this case make the simple experiment of compounding free and forced vertical velocity for the height of 1000 meters above the ground. The result is given on plate XLI. When this chart is compared with the simultaneous data regarding the distribution of precipitation, of cloudiness, and of blue sky, a considerable accordance will be seen to exist in spite of the great extrapolation involved in the estimation of vertical velocities at so great heights from observations taken only at the ground.

202. **Practical Applications of the Charts of Motion.**—It will involve no difficulty to introduce the drawing of charts like that of plate XXXVIII, representing the horizontal motion by lines of flow and curves of equal intensity, into the daily meteorological service for the forecast of the weather. When the chart is to be drawn only for qualitative purposes, it will not be required to use the more circumstantial method to draw first the isogonal curves. The main course of the lines of flow can be sketched directly. When the drawing of the lines of flow and of the curves of equal intensity is distributed between two workers, and these have acquired some experience, it will cause them no difficulty to have the chart of motion ready in a space of time comparable to that required for drawing the common charts representing pressure, temperature, and other data.

These charts of motion possess many characteristic features in the form of singularities which are in an obvious relation to the conditions of the weather. Therefore we have reason to believe that experience will gradually lead to practical rules for weather-forecasts based upon the examination of the charts of motion in themselves or in connection with the other charts.

When the draftsman has acquired sufficient experience, a rapid examination of the chart of horizontal motion will show him the places for the strongest forced vertical motion and for the strongest free one. By making a few measurements in these places he will be able rapidly to sketch charts of the vertical motion, and there is hardly any reason to doubt that these charts would prove useful for the forecast of precipitation.

The charts of motion may also be useful for aerial navigation. For instance, a glance at the chart of plate XXXVIII will show at once that an air-ship which moves, *e. g.*, 15 meters per second will not be able to go in a straight line say from Bismarck

in North Dakota to the southern coast of Lake Superior, for here it would have a head-wind of 28 meters. But it would easily accomplish the voyage by the circuit south of the center of the cyclone. If we were able to estimate the degree of persistency of the state of motion, and the direction in which the changes are to take place, it would be possible by use of such charts to plan the course of aerial ships so that they will reach their destination in the shortest time.

203. **Charts of Acceleration.**—Charts XXXVII to XLI exemplify the kinematic diagnosis as far as it can be carried out by use of observations taken at one epoch only, and only at the stations at the ground. When we have observations from two successive epochs, we can go one step farther and determine the acceleration of the motion kinematically.

The charts of plates XXXVII or XXXVIII were taken from the registered values of wind-intensities and wind-directions during the hour from 7 to 8 a. m., time of 75th meridian. The charts of plates XLII and XLIII show the corresponding representation of the motion derived from the values registered during the hour from 10 to 11. As will be seen, the point of convergence has been displaced a little more than 200 kilometers toward northeast, but otherwise the general features of the chart are unchanged.

In order to determine the average acceleration during the interval of time between the two epochs, we first form the chart of the average velocity for this interval of time. This is done by addition and division by 2 of the two vector-fields represented by plates XXXVII and XLII. The result as obtained directly, represented by isogonal curves and curves of equal intensity, is shown on plate XLIV; plate XLV shows the corresponding representation by lines of flow and curves of equal intensity.

By the subtraction of the same two vector-fields and division by the interval of time, 3 hours or 10,800 seconds, we form the chart of *local* acceleration. Plate XLVI contains the representation of this vector by isogonal curves and curves of equal intensity, and plate XLVII gives the representation by vector-lines and curves of equal intensity.

From charts of average motion (plates XLIV or XLV) we derive the chart of stationary acceleration as described in section 197 (A). The result is given on plate XLVIII by isogons and intensity-curves, and on plate XLIX by vector-lines and intensity-curves.

The true acceleration of the moving particles is finally obtained by the addition of the vector-fields representing local and stationary acceleration. The result is represented by the charts of plates L and LI, on the first by isogonal curves and intensity-curves, on the second by vector-lines and intensity-curves.

Much more experience than we have at present must be gained before we can estimate the degree of objective reliability of a chart of acceleration like that given on these plates. In the western mountainous parts, where in many cases great doubt may arise as regards the charts of velocity from which the chart of acceleration has been derived, the values found for the acceleration must of course be used with

great reserve. The same should be the case along the borders of the chart. But in the more central part, in the Mississippi valley, we have every reason to believe that the chart gives a good approximation to the truth. The question of attaining the same reliability of the chart of acceleration in the other districts will, as will be understood at once, simply be a question of further developing the net of stations and improving the methods of observing the wind.

204. Main Example of Kinematic Diagnosis, Europe, 1907, July 25, 7 a. m. Greenwich.—In the preceding examples we have exclusively used observations from the common meteorological stations at the ground. We shall now consider a case where observations, though in quite insufficient number, are at hand also from the higher strata, namely, the aerological observations on the morning of July 25, 1907.

To begin with the observations from the ground, plate LII gives the distribution of pressure and of mass in the lowest atmospheric sheet. The single lines give the absolute topography of the 1000 m-bar surface and the double lines the relative topography of the 900 m-bar surface. It will be seen that pressure is rather uniformly distributed. A relatively high ridge goes from Iceland over Scotland and the North Sea toward the Balkan Peninsula. East of this ridge a great number of local maxima and minima are seen. Plate LIII gives the winds observed, represented by arrows, by numbers of direction, and by numbers of intensity. The winds are generally faint and irregularly distributed.

Plate LIV gives the corresponding continuous representation by use of isogonal curves and curves of equal intensity. The diagram of isogons shows a great number of positive and negative singular points. Plate LV gives the representation by lines of flow and intensity-curves, with the corresponding great number of neutral points as well as of points and lines of convergence and of divergence. The comparison with the chart of pressure plate LII shows that the points of convergence with great regularity coincide with the small depressions, and the points of divergence coincide with the corresponding heightening of the isobaric surface of 1000 m-bar pressure.

The chart of plate LIV is drawn upon a blank which represents the average pressure at the ground (see plate XXX). From this chart we therefore easily derive that of vertical specific momentum at the ground, plate LVI.

When we pass to air-motion in the higher strata, we must limit our considerations to the small area where we have the closest network of aerological stations. We have done this on plates LVII to LX, which correspond respectively to the standard sheets X, IX, VIII, and VII (section 107). The incomplete sheet XI is so thin that we have left it out of consideration. The wind-observations are represented on the charts A of these plates by arrows and intensity-numbers, as we have mentioned already. As the arrows and numbers are too few in number for the drawing of the charts of horizontal motion, we have used in anticipation dynamic principles for giving at least a tolerably probable reconstruction of the horizontal motions as developed in section 139. The reconstructed horizontal motions are represented by lines of flow and curves of equal intensity on the charts B of plates LVII to LX. Of course we must leave open the question regarding the degree to which we have thus

succeeded in reconstructing the true horizontal motions. We shall therefore use them only to illustrate the *formal* methods of a complete kinematic diagnosis of atmospheric motions.

From the charts B we then derive auxiliary charts representing the contribution of each sheet to the vertical component of the specific momentum. These auxiliary charts have not been reproduced. But the method of drawing them is that developed in section 187 (C), and for the incomplete parts of the sheets in section 189.

As soon as these auxiliary charts are drawn we find the true vertical specific momenta at different standard isobaric surfaces by successive graphical additions

By graphical addition of the vertical specific momentum at the ground (plate LVI) to that produced in the incomplete sheet X we get the chart LVII c, which gives the vertical specific momentum at the isobaric surface of 900 m-bar pressure.

By graphical addition of this vertical specific momentum to that produced in the sheet IX we get the chart LVIII c, which represents vertical specific momentum at the standard isobaric surface of 800 m-bar pressure.

By graphical addition of this vertical specific momentum to that produced in sheet VIII we get in the same manner the chart LIX c, representing the vertical specific momentum at the standard isobaric surface of 700 m-bar pressure.

By graphical addition of this vertical specific momentum and that produced in sheet VII we get the chart LX c, which represents the vertical specific momentum at the standard isobaric surface of 600 m-bar pressure.

The plates LVII to LX thus give the complete result of the static and the kinematic diagnosis of atmospheric conditions on July 25, 1907, about 7 a. m., Greenwich time, on the basis of the aerological soundings performed about this time. The charts LVII A to LX A give the field of pressure and of mass as the result of the static diagnosis; the charts LVII B to LX B give the horizontal motion within each of the four isobaric sheets defined by the charts LVII A to LX A. The charts LVII c to LX c represent the vertical transfer of mass from one to the other of these isobaric sheets.

We have exemplified the diagnosis by taking the four lowest atmospheric sheets. Some of the observations on the day under consideration go much higher. But it will have no interest to extend the diagnostic work further before the observations have attained the completeness required for working out diagnoses which have an unquestionable objective value. Till then we can only exemplify the formal methods.

If the observations were performed according to the plan which we have developed in Chapter I, we should be able to work out complete diagnoses at epochs which were sufficiently near each other to allow us to derive also the fields of acceleration within all atmospheric sheets. This would be the first step in opening the way for serious investigations in atmospheric dynamics.